THE THIRTY-SECOND SUMMER MEETING OF THE SOCIETY

The thirty-second summer meeting of the American Mathematical Society was held at Ohio State University, Columbus, Ohio, on Wednesday and Thursday, September 8-9, 1926, immediately preceded by the summer meeting of the Mathematical Association of America. A joint session of the two organizations was held on Wednesday morning, at which addresses were delivered by Professor E. W. Chittenden on *The metrization problem and related problems in the theory of abstract sets*, and Professor E. T. Bell on *Successive generalizations in the theory of numbers*. Professor Chittenden's address will be published in an early issue of this BULLETIN; Professor Bell's will appear in the AMERICAN MATHEMATICAL MONTHLY. There was a joint dinner of the two organizations on Wednesday evening, at the Faculty Club of the University. At the close of the sessions a resolution was passed expressing the appreciation of the Society for the very cordial hospitality of the University, which made this meeting one of the pleasantest in its history. The authorities of the Ohio State University contributed greatly to the success of this meeting by putting the dormitories, Mack Hall and Oxley Hall, at the disposal of the Society; the reception held on Tuesday evening in the Faculty Club was a very pleasant occasion. The cordiality of the officials of the University and the unremitting zeal of the members of its Department of Mathematics to promote the success of the meeting will be gratefully remembered by all those who were present.

There was a large attendance, probably the largest for any summer meeting at which no colloquium was held. It included the following one hundred seventeen members of the Society.

Akers, W. E. Anderson, Anning, Arnold, Bareis, Barnett, E. R. Beckwith, E. T. Bell, H. A. Bender, S. R. Benedict, A. A. Bennett, Blumberg,

Borger, Bowles, Bussey, Cairns, C. C. Camp, J. W. Campbell, Candy, Caris, Carmichael, Chang, Chittenden, E. H. Clarke, Abraham Cohen, L. P. Copeland, Cramblet, Crane, Dantzig, Davisson, J. M. Davis, Dresden, Dushnik, Emch, G. C. Evans, Everett, Farnum, Feinler, Peter Field, Finkel, Fite, Foraker, W. B. Ford, Fry, C. A. Garabedian, Garretson, D. C. Gillespie, Glenn, B. C. Glover, J. W. Glover, Gummer, Harkin, E. R. Hedrick, Hollcroft, Ingraham, Dunham Jackson, Kempner, Kuhn, LeStourgeon, A. D. Lewis, F. A. Lewis, Lindemann, MacDuffee, T. E. Mason, A. S. Merrill, Michal, Michie, G. A. Miller, I. L. Miller, Mills, Miser, C. L. E. Moore, C. C. Morris, F. H. Murray, Newlin, Olson, F. W. Owens, Pennell, Rasor, Raynor, R. G. D. Richardson, Rider, H. L. Rietz, J. C. Rietz, E. D. Roe, J. R. Roe, Rowland, Seely, Shewhart, Shohat, W. G. Simon, Sisam, Slaught, Smail, C. E. Smith, Spenceley, Stafford, Swartzel, Teach, T. Y. Thomas, Tinner, Tripp, B. M. Turner, Vandiver, T. O. Walton, J. H. Weaver, R. A. Wells, Westfall, R. L. Wilder, Wiley, C. O. Williamson, F. M. Wright, Yanney, Yeaton, J. W. Young, Yowell, Zehring.

The Council announced the election to patron membership of

The Bell Telephone Laboratories;

and to sustaining membership of

Mr. Adolph S. Ochs.

It was announced that since the April meeting of the Council and at this meeting the following thirty-three persons were elected to ordinary membership in the Society.

Miss Nola Lee Anderson, Saint Catharine, Missouri;
Mr. Thomas C. Benton, University of Pennsylvania;
Mr. William J. Berry, University of Colorado;
Mr. Edward Bright, Sandwich, Cape Cod, Mass.;
Mr. John Clark Brixey, University of Oklahoma;
Mr. Bailey LeFevre Brown, Princeton University;
Mr. William Bucke Campbell, Cornell University;
Professor Harry Clyde Carver, University of Michigan;
Mr. Frank Hsi-Lu Chang, University of Chicago;
Dr. Edgar Dehn, Wagner College, Staten Island, New York;
Dr. Frederick William Doermann, New York University;
Mr. Howard P. Doole, University of Wisconsin;
Mr. Leamon A. Dye, University of Rochester;
Miss Fay Farnum, Cornell University;
Professor John Wildeboor Hurst, Montana State College;
Mr. Maurice L. Hartung, University of Wisconsin;
Professor William Scribner Kimball, Michigan State College;
Professor Jean Martial Lapeyre, Loyola University, New Orleans;
Mr. Derrick Henry Lehmer, University of California;

Mr. Clarence Isaac Lubin, University of Cincinnati;
Mr. Charles Thomas Male, Union College;
Mr. Alexander Maslow, University of Michigan;
Mr. John M. Melchiors, 510 Sixth St., Kaukauna, Wis.;
Mr. Henry Otten, Jr., Interborough Rapid Transit Co., New York, N. Y.
Mr. Harry S. Pollard, University of Wisconsin;
Mr. Harold Conway Shaub, Cornell University;
Professor James Abram Garfield Shirk, Kansas State Teachers College;
Professor Emory Potter Starke, Rutgers University;
Mr. Earl Edward Steinert, Union College;
Mr. George Clarence Vedova, Columbia University;
Professor Lloyd A. H. Warren, University of Manitoba;
Mr. Gordon T. Whyburn, University of Texas;
Mr. John Williamson, University of Chicago.

The following person was elected to membership as nominee of the New York Edison Company:

Mr. J. W. Lieb.

The secretary announced that the following persons had taken membership in the Society under the reciprocity agreement with the London Mathematical Society:

Professor T. Kubota, Tôhoku Imperial University;
Dr. O. Szasz, Frankfurt a/M., Germany.

Eighteen applications for membership were received.

It was announced that the following persons had been appointed to represent the Society: on the Sub-Committee on Mathematics and Mathematical Signs of the Sectional Committee on Scientific and Engineering Symbols and Abbreviations of the American Engineering Standards Committee, Professors Abraham Cohen and Oswald Veblen; to represent the Society at the Seventy-Fifth Anniversary of the founding of Milwaukee College on June 12, 1926, Professor Arnold Dresden; to represent the Society at the Semi-Centennial of Grove City College on June 15–16, 1926, Professor O. P. Akers; to represent the Society at the Centennial of Western Reserve University on November 12–13, 1926, Professor W. G. Simon; to represent the Society at the inauguration of President A. B. Hall at the University of Oregon on October 18–23, Professor F. L. Griffin. Also that a Committee on policy had been appointed to consist of

Professors Bliss, Eisenhart, Huntington, Dunham Jackson, and R. L. Moore.

Further announcements included that President Birkhoff's retiring address to be delivered at the annual meeting in Philadelphia is entitled *A mathematical critique of some physical theories*; that the 1927 meetings of the Society in New York have been set for the last Saturday in February, the last Saturday in October, and the first Saturday in May; that the Summer Meeting and Colloquium for 1927 are to be held at Madison, Wisconsin, during the week beginning September 4.

The Council accepted the invitations of Amherst College and of the University of Colorado to hold the summer meetings of the Society at these institutions in 1928 and 1929 respectively.

At the joint session on Wednesday forenoon the following resolution was adopted:

"*Resolved*, That the American Mathematical Society and the Mathematical Association of America, in joint session at Columbus on September 8, 1926, express their deep sense of loss in the death of Professor Bohannan, on June 20, 1926, and their profound regret that the meetings on the campus so long animated by his personality could not be graced by his presence; and direct that copies of this resolution be transmitted to Mrs. Bohannan and to the President of the Ohio State University."

The following resolution was adopted at the session of Wednesday afternoon:

"On recommendation of its Council, the American Mathematical Society, in session at Columbus, Ohio, on September 9, 1926, expresses its deep sense of loss in the death of Frank Nelson Cole on May 26, 1926. He was for many years the Society's most active executive officer. From an early date in its history until 1920, when he passed his duties on to others, he ably guided the development of the Society. As Secretary from 1895, as a member of the Editorial Board of the BULLETIN from 1898, and as its Editor-in-Chief

from 1900, he led the Society from its modest beginnings to a state of solid accomplishment. He exercised his functions with a skill which excited admiration and which gave the American Mathematical Society an established place in the scientific world. When he retired, he could turn over to his successors a healthy structure, which was able to withstand the stresses of the very difficult post-war period. The Society wishes also to place on record at this time its grateful recognition of his devoted service to the ideals of American mathematical science. His memory will remain an inspiration to all who may in the future serve the interests of the Society and the cause of mathematics in America."

Professor G. C. Evans, vice-president of the Society, presided at the beginning of the joint session on Wednesday morning, relieved by Professor Dunham Jackson, president of the Association. The later sessions of the Society were presided over by Vice-Presidents Evans and E. T. Bell and Professor R. D. Carmichael.

Titles and abstracts of the papers read at the regular sessions follow below. The papers of Aldrich, Altshiller-Court, Dickson, Ettlinger, Evans, Gehman, Griffiths, Hazlett, Hill, Kellogg, Koopman, Michal, Norman Miller, Morse, Murnaghan, Neelley, Niemytski, Rainich, Rees, Robinson, Sierpinski, Snyder, Stone, Sturdivant, J. M. Thomas, T. Y. Thomas and Michal, Vandiver (second paper), Vandiver and Wahlin, Vinogradov, Walsh, G. T. Whyburn, W. M. Whyburn and C. E. Wilder were read by title. Mr. Mullings, Mr. Rees, and Mr. Sturdivant were introduced by Professor Ettlinger, Professor Vinogradov by Professor H. H. Mitchell, and Professors Niemytski and Sierpinski by Professor Chittenden.

1. Professor J. H. Weaver: *Invariants of a poristic system of triangles.*

Gallatley in his book *Modern Geometry of the Triangle* discusses certain properties of a system of triangles having a fixed incircle and a fixed circumcircle. However he does not attempt to find what curves associated

with the system remain invariant. The author shows that there is a single invariant line associated with the system of triangles, namely, the line determined by the points where the bisectors of the exterior angles meet the opposite sides. He also shows that there is a one-parameter family of invariant curves of the second degree associated with the system.

2. Professor T. R. Hollcroft: *Consecutive multiple points of algebraic curves.*

Of the total number of conditions on an algebraic curve accounted for by a given series of consecutive multiple points, some are projective and some metric. The number of conditions of each kind can be found when the constituents and the manner of formation of the singularity are known. In the determination of a curve, projective conditions belonging to independent singularities are always independent and necessary while metric conditions associated with any singularity are not necessary and any or all of them may be dependent. For a given singularity, it is sufficient but not necessary that the multiple points which became consecutive to build up this singularity be the same as those into which it can be resolved. There is a definite number of projective as well as a definite number of metric conditions belonging to a given singularity. The number of projective conditions actually given by a singularity is always this number however the singularity may have been formed, but the number of metric conditions varies with the manner of formation. Limits are derived for the number of consecutive multiple points of any order, with or without coincident branches, on curves of any order.

3. Professor F. A. Foraker: *Nepolar locus of a point with respect to a conic.*

If A, P, B, and Q are four points on a line which satisfy the condition $(AP/PB)\cdot(AQ/QB)=-1$ they will be defined as a neharmonic range. The nepolar locus of a point with respect to a conic is the locus of the neharmonic conjugate of the given point with respect to the two points in which any secant through the given point cuts the conic. The nepolar locus of the point (x_1, y_1) with respect to the conic $F(x, y)=ax^2+2hxy+by^2+2gx+2fy+c=0$ is the cubic $FP-2FF_1+F_1P=0$, where $p=axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c$. It is shown that the nepolar locus of a point with respect to a conic passes through the points of intersection of the conic and its polar with respect to the conic. Also, that the nepolar locus of a point on a conic with respect to that conic itself consists of the conic and its tangent at the given point. A number of graphs are shown exhibiting a variety of cubics derived from the equation.

4. Professor G. Y. Rainich: *On singularities of vector fields in space.*

A configuration consisting of a closed surface with a vector in every point of it is studied from the topological point of view. A reduction to

a normal form represented by a closed surface with a finite number of contours is given. The application of this reduction to the question of topological invariants of a singularity of a vector field in three-space is considered.

5. Professor C. L. E. Moore: *Note on surfaces in a non-Riemannian space.*

This paper discusses some of the properties of surfaces in a general space and especially of those whose coordinates satisfy a linear contravariant differential equation.

6. Professor Arnold Emch: *On triple systems and their connection with certain problems of algebraic geometry and analysis situs.*

Since the time when Steiner first drew attention to the possibility of so-called triple systems, combinatory and group properties of such systems have been investigated rather extensively by a number of mathematicians. The purpose of this paper is to show how the geometric aspect of the problem reveals an interesting connection between systems of pairs of triple systems without a common triad and a class of algebraic surfaces passing singly or multiply through each of the $(1/2)n(n-1)$ joins of n generic points in euclidean or projective 3-space. It is further shown that from the standpoint of analysis situs two such triple systems are equivalent with either one- or two-sided complexes of 2-cells of a certain genus, which in every case may easily be determined. As such complexes are triangulated from the start, the problem of their mapping on plane polygons offers no particular difficulty, as is shown by a number of examples.

7. Professor Virgil Snyder: *On a problem in closure.*

This paper will appear in full in an early issue of this BULLETIN.

8. Professor N. Altshiller-Court: *On the de Longchamps circle of the triangle.*

C. de Longchamps considered the circle, now bearing his name, orthogonal to the three circles (A, a), (B, b), (C, c), described from the vertices of a triangle with radii equal to the opposite sides (JOURNAL DE MATHÉMATIQUES SPÉCIALES, 1886, pp. 57 etc.). The present paper considers mainly the circles of similitude and the bisecting circles of these circles taken in pairs. The circles of similitude are the symmetrics of the circles of Apollonius of the basic triangle with respect to the perpendicular bisectors of the corresponding sides. If the angles of the triangle are acute, the three circles of similitude have two real points in common, each of which has its distances from the vertices of the triangle proportional to the opposite sides; and these two points are the only points of the plane having this property. They lie on the Euler line of

the triangle and are the limiting points of the coaxial system of circles determined by the de Longchamps circle, the circumcircle, and the nine-point circle of the basic triangle.

9. Professor T. Y. Thomas and Dr. A. D. Michal: *Concerning differential invariants in certain new differential geometries.* Preliminary communication.

The authors study differential invariants in connection with the theory of affinely connected manifolds of the geometry of paths and with the theory of relative quadratic differential forms, by means of the differential equations defining the invariants. Infinitesimal transformations of tensors and r-parameter group considerations enter in the discussion. An attack is made on equivalence problems by means of a new concept introduced in connection with the determination of the number of independent invariants. It is intended to apply and extend the methods to the treatment of projective differential invariants.

10. Mr. J. H. Neelley: *The invariant conditions for some compound singularities of the plane rational quartic.*

Using invariants, as given by R. M. Winger in the AMERICAN JOURNAL OF MATHEMATICS, vol. 36, this paper develops invariant conditions for the plane rational quartic curve to have a cusp of the second kind, sometimes called a ramphoid cusp. The invariant condition for the compound singularity due to the coincidence of three nodes in a manner to form an oscnode have also been obtained by a very neat method of representation which is unique to this type of curve.

11. Professor J. W. Young: *On a form of general geometry.* Preliminary communication.

In a paper on the geometries associated with a certain system of Cremona groups, TRANSACTIONS OF THIS SOCIETY, vol. 17, p. 233, the author, in collaboration with F. M. Morgan, showed that a certain system of Cremona groups in a space S_n, $n=1, 2, 3, \cdots$, could be represented by means of a linear associative algebra on the points of S_n in the form $x'=(ax+b)/(cx+d)$. This representation enabled them to derive certain fundamental theorems of the geometries associated with these groups by using elementary properties of the projective group on a line. The present paper derives similar theorems in a much more general setting. Given a class S of elements $a, b, c, \cdots$ (which may be points, or lines, etc., but need not be geometric) and a linear associative algebra A on the elements of S (a previous paper, not yet published, deals with the construction of such algebras). If multiplication is commutative in A, theorems, analogous to those referred to above, are readily derived for the geometry associated with the linear fractional group on a single variable in A. The paper also makes a beginning toward extending its methods to algebras on S in which multiplication is not commutative.

12. Professor J. A. Shohat: *On certain applications of Tchebycheff polynomials in several variables.*

This paper solves the following problem: Given an arbitrary polynomial $G_n(x_1, \cdots, x_s)$ of degree $\leqq n$, in s independent variables, find the maximum of the absolute value of an arbitrary given homogeneous polynomial of degree $m(=1, 2, 3, \cdots)$ of the coefficients of G_n if we know the value of $\int_D G_n^2(x_1, \cdots, x_s)\, q(x_1, \cdots, x_s)dx_1 \cdots dx_s$, or of max. $|q(x_1, \cdots, x_s)\, G_n(x_1, \cdots, x_s)|$ in D, where D is a certain s-dimensional domain, finite or infinite, and q is defined in D. The method used here is an extension of that given by the author in a paper presented to the Society in September, 1925. We utilize Tchebycheff polynomials in several variables. Numerous applications to Tchebycheff polynomials and to polynomials in general are obtained.

13. Professor H. L. Olson: *Linear congruences in quaternion arithmetic.*

This paper extends to the arithmetic of rational integral quaternions the concept of a linear congruence. It presents a general process of solution and a criterion for the solvability of a linear quaternionic congruence.

14. Professor H. L. Olson: *Linear congruences in a linear algebra.*

This paper presents the condition for the existence of a solution of a linear congruence $axb \overline{\underline{+}} c \pmod{m}$ or $axb \overline{\underline{\equiv}} c \pmod{m}$ in any rational semi-simple linear algebra of finite order. In the case in which the norms of a and b are relatively prime to the norm of m, the solution is exhibited in explicit form.

15. Professor G. A. Miller: *Subgroups of index p^2 contained in a group of order p^m.*

The author proves that the number of the invariant subgroups contained in a group G of order p^m which correspond to cyclic quotient groups of order p^2 is of the form $p^{\alpha-1}(1+p+p^2+\cdots)$ where α is the number of the independent generators of G, whenever there is at least one such invariant subgroup in G; also, that if a group of order p^m contains non-invariant subgroups of index p^2, then every set of such conjugate subgroups generates a subgroup of index p which involves at least one set of p conjugate subgroups of index p^2 whose common cross-cut is of order p^{m-3} and is invariant under the entire group.

16. Professor G. A. Miller: *The so-called Naperian logarithms.*

This paper calls attention to the fact that the numbers which Napier called logarithms do not satisfy the fundamental law on the logarithm of the product of two numbers. It is also noted that the equation Nap.

$\log y = 10^7 \log_e (10^7/y)$ which is commonly found in our histories of mathematics is inexact. In his well known lecture on *John Napier and the invention of logarithms*, 1614, E. W. Hobson refers to this equation as expressing "the exact relation" if Nap. log y represents the logarithm of y in accordance with Napier's tables. It is observed that this conflicts with Lindemann's theorem that in the equation $y = e^x$, x and y are both algebraic only when $x = 0$. The author concludes that it is undesirable to call logarithms to the base e Naperian logarithms. The base e was first used as a definite number long after Napier and Speidell published their tables. Statements relating to the development of logarithms are frequently obscured by a lack of distinction between exact relations and approximate ones.

17. Dr. H. A. Bender: *Concerning an important theorem in group theory.*

The theorem concerning simply isomorphic groups is one of the most important theorems in group theory, yet with the present development of the theory of groups it is practically useless for determining all the distinct groups of a given class. This is shown by several examples; illustrations are also given of the misuse of this theorem. Suggestions are given for the development of certain phases of the theory of groups which will aid materially in the application of the above theorem.

18. Professor O. E. Glenn: *A symbolical modular theory of biternary concomitants.*

The third lemma in the group of theorems preliminary to Gordan's third proof of Gordan's theorem finds its analog, in this paper, for the case of biternary transvectants $(f, g)_k^{ij}$, as follows: If a finite system (A) of connexes, $A_1, A_2, \cdots$, all of which are concomitants of $f = a_x^m \alpha_u^n$, includes f and is relatively complete for a set (G) of constant symbolical moduli $G_1, G_2, \cdots$, and if a finite system (B) of connexes $B_1, B_2, \cdots$, which are also concomitants of f, is also relatively complete for a set (Γ) of moduli $\Gamma_1, \Gamma_2, \cdots$ and contains one form for each modulus $G_1, G_2, \cdots$ whose only symbolic factor is precisely that modulus, then the system (C) derived by transvection between (A) and (B) is relatively finite and complete modulo (Γ). The moduli are of the various forms $a_\alpha{}^r a_\beta{}^s \cdots b_\gamma{}^t \cdots$, $a_\alpha{}^q a_\beta{}^r \cdots b_\gamma{}^s (abc)^t$, $(abc)^r, \cdots$. With this lemma a succession of theorems furnishing methods for the generation of fundamental systems of connexes is completed. The known system of $a_x \alpha_u$ (Clebsch and Gordan) is now found with surprising brevity; the extensive system of $a_x^2 \alpha_u$ is obtained by very direct processes. Further investigation is being made concerning the uses of Gordan's lemma and its extensions, in the symbolic modular theory of biternary forms. The generalization has been made by Wood, from the known binary theory, to the case of covariantive ternary perpetuants.

19. Professor C. C. MacDuffee: *Ideals in linear associative algebras.*

This paper is an introduction to the study of ideals in linear associative algebras, using Dickson's definition of set of integral elements. It is shown that if every left ideal is principal, every pair of integral numbers has a greatest common right divisor, and conversely. A greatest common right divisor is unique up to a unit left factor if its norm is not zero. The corresponding theorems hold for greatest common left divisors. The property of unique factorization in a non-commutative algebra does not always follow from the existence of the greatest common divisors. Dedekind's ideals are of little use in this problem, since $\alpha=\beta\gamma$ does not always imply $\{\alpha\}=\{\beta\}\{\gamma\}$. This difficulty is avoided by introducing ideals of another sort, namely, by defining $\{\alpha\}$ to be the first matrix R_α of α. Since matrices with integral elements are factorable into primes in a manner which is unique except for unit factors, these ideals restore unique factorization to every rational semi-simple algebra.

20. Professor L. E. Dickson: *All integral solutions of* $ax^2+bxy+cy^2=w_1 w_2 \cdots w_n$.

This paper appears in full in the present issue of this BULLETIN.

21. Professor H. S. Vandiver: *A necessary and sufficient condition that an integer be prime.*

The theory of definite quadratic forms is used to establish the following theorem: Consider all the sets of integers (x, y) which satisfy $x^2+ay^2=kn$, where a and n are given integers, $a<n$ and not a square, a prime to n; an integer v exists such that $v^2+a\equiv 0 \pmod{n}$; k is an integer not having a square factor different from unity dividing y^2, such that $k\leq 2(a/3)^{1/2}$, and x and y are positive integers or zero. Then the integer n is prime if and only if there is a unique set (x', y') satisfying the above conditions and such that x' and y' are each prime to n. The criterion may be employed with advantage when n is large.

22. Professor H. S. Vandiver: *On the theory of finite rings.*

In a paper in the TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), p. 293, the writer considered a theory of finite algebras, defined as systems consisting of a finite number of elements which combine according to the laws of elementary algebra, except that division is not always possible and when possible is not always unique. In accordance with present day terminology such a system is called a ring. Kircher showed (AMERICAN JOURNAL, vol. 39 (1917), p. 279) that non-units in a ring did not always decompose uniquely into prime non-unit factors. The present paper investigates this question further. Two elements whose difference is a unit are called *absolutely distinct.* If, in comparing two factorizations of a non-unit element into prime non-unit factors we regard as distinct only those

elements which are absolutely distinct, the writer shows that for an extensive class of rings this type of decomposition is unique, and conjectures that the theorem is general.

23. Professors H. S. Vandiver and G. E. Wahlin: *Report on certain topics in the theory of algebraic numbers.*

The aim of this report is to supplement the report on the theory of algebraic numbers of the Committee on Algebraic Numbers of the National Research Council published in the BULLETIN OF THE NATIONAL RESEARCH COUNCIL, Feb. 1923. The class number in any algebraic field, irregular cyclotomic fields and the application of cyclotomic fields to Fermat's last theorem, are the topics covered in the present report; the development of each of these subjects is traced from the beginning of its history. The report will be published by the National Research Council.

24. Professor O. C. Hazlett: *Formal modular protomorphs.*

This paper proves the principal theorems of a recent paper by W. L. G. Williams, TRANSACTIONS OF THIS SOCIETY, vol. 28 (1926), p. 183, under much weaker hypotheses than those used by him. He proved that if $GF[p^n]$ is any Galois field of order p^n, and if $f(a; x)$ is any binary form for which no one of the binomial coefficients is congruent to zero, modulo p, then a complete set of formal modular protomorphs of f with respect to $GF[p^n]$ consists of a complete set of algebraic protomorphs of f together with $a_0 a_1 p^n - a_0 p^n a_1$. Similarly for a system of forms. The present paper proves these theorems under the hypothesis that f is such that its order is not congruent to 0 modulo p.

25. Professor J. W. Young: *On the partitions of a group.*

A partition of any group G is a class $[H]$ of subgroups of G such that every element $(\neq 1)$ of G is in just one H. Let $G_{1,s}$ be the group generated by any element $s(\neq 1)$ of G, and let $G_{k+1,s}$ be the group generated by all the $G_{k,t}$ containing $s(k=1, 2, 3, \cdots)$. If there exists a smallest n (finite or transfinite) such that for every s of G, $G_{n,s}=G_{n+1,s}$, G is said to be of type n. If G is of type n the class of distinct subgroups $[G_{n,s}]$ as s ranges over G forms a partition, and conversely. The partition obtained in this way is called primitive. A partition consisting of two or more distinct subgroups is called proper. Any abelian group is of type $\leqq 4$; if it contains no elements of finite order, of type $\leqq 2$. The primitive partition of an abelian group G is proper, if and only if, either (1) G contains no elements of finite order, or (2) all the elements of G are of the same prime order. A partition H of a group G is said to be an L-partition if the group K generated by any finite set $H_1, H_2, \cdots, H_k$ of the partition is such that any other H_t either is entirely contained in K or has no element $(\neq 1)$ in common with K. Other theorems relate to the conditions on G under which the primitive partition of a group is an L-partition.

26. Professor J. W. Young: *A new formulation for general algebra.* Second paper.

This paper gives a simpler and more symmetric set of properties on which to base the formulation given in a previous paper with the same title (this BULLETIN, vol. 19, pp. 514, 515).

27. Professor M. H. Ingraham: *The infinite case of the equivalence of linear associative algebras to matrix algebras.* Preliminary report.

If an algebra A over a field F has a finitely linearly independent base in terms of which all its elements may be expressed as finite linear combinations with coefficients in F, then A is equivalent to an algebra of matrices, infinite in order if A is infinite in order, with elements in the field and such that each column contains only a finite number of non-zero elements. A is reciprocal to an algebra of matrices of the same type. Such a base exists if A has a well-ordered base, not necessarily linearly independent. Certain classifications of elements and sub-algebras are given. The problem of the extension of the notion of the minimum equation of an element and the definition of integral elements is discussed.

28. Mr. M. E. Mullings: *The rotational derivative and some applications.*

This paper gives a definition of a rotational derivative of a vector function analogous to the directional derivative of a vector or scalar function (see Gans, *Einführung in die Vektoranalysis*, Leipzig, 1921, p. 34). For a simple surface and a curve of type C, two important auxiliary theorems are obtained. By use of his definition, the Duhamel-Moore theorem, and the two auxiliary theorems, the author gives a completely satisfactory proof of Stokes' theorem which is simpler and more direct than previous proofs.

29. Dr. C. C. Camp: *An expansion connected with a partial differential equation.* Preliminary communication.

There will be suitable characteristic solutions of $\partial u/\partial x+\partial u/\partial y+(\lambda f+g)u=0$ and the boundary conditions $u(x, \pi)=u(x, -\pi)$, $u(\pi, y)=u(-\pi, y)$, if f and g are each composed of terms in x and y alone, besides one of the form $DA(x, y)\equiv(\partial/\partial x+\partial/\partial y)A(x, y)$, provided A also satisfies the boundary conditions. Such a function in f or g does not influence the values of λ. By the transformation $u=u_1e^{-\lambda A-B}$, the equation may be reduced to a simpler form. By writing $u=X(x)Y(y)$, this can then be broken up into a system of ordinary equations which are new because of the way in which the parameters λ, μ enter. By proper choice of new parameters, the author builds up the analog of the Green's function; he investigates the convergence of the expansion of an arbitrary function $F(x, y)$ by using an extension of the contour integral method of Birkhoff.

This treatment seems to exhaust the possibilities for the given system although the theory may be extended to n variables and to less simple boundary conditions.

30. Professor O. D. Kellogg: *On some theorems of Bôcher concerning isolated singularities of harmonic functions.*

This paper appears in full in the present number of this BULLETIN.

31. Mr. L. B. Robinson: *On equations in mixed differences; continuation.*

In this paper (see this BULLETIN, vol. 32, p. 316) the author considers the equation $u'(x)=\sum_{i=1}^{n}[P_i(x)/\{x(x+r)\}]u[a_ix+\delta_i]+Q(x)/\{x(x+r)\}$, and shows that if three inequalities are satisfied a single infinity of solutions exists, each with a logarithmic singularity at $x=0$ and $x+r=0$.

32. Professor P. R. Rider: *A figuratrix for double integrals.* Second paper.

In the first paper by this title the author defined a figuratrix for double integrals in the calculus of variations and showed that a certain necessary condition obtained by Kobb implied that the total curvature of the figuratrix must be positive. In this paper he gives geometric interpretations of the transversality condition and the E-function.

33. Professor C. A. Garabedian: *Disk of constant or variable thickness in uniform or accelerated rotation.*

By applying to the problem of the rotating disk the method of series elaborated in the author's *Circular plates of constant or variable thickness* (TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), p. 343), it is possible to bring under a uniform treatment some aspects of the subject hitherto studied separately. When the thickness is constant, the results for uniform rotation include certain solutions due to Chree, while the results for accelerated rotation show that the formulas recently derived by Prudon and by Pigeaud on the assumption of a thin disk are equally applicable to the thick disk. In the case of a disk of variable thickness in uniform rotation, it turns out that the leading term in the solution in series is given by the Stodola differential equation; subsequent terms in the series offer a means of judging the accuracy of the Stodola approximation. Similar results are obtained for the disk of variable thickness in accelerated rotation, and here the differential equation analogous to that of Stodola appears to be new. An abstract will be found in the COMPTES RENDUS, vol. 183, Aug. 17, 1926, p. 416.

34. Professor F. D. Murnaghan: *The Boltzmann-Hopkinson principle of superposition in Maxwell's theory of the layer dielectric.*

In this paper, which is a continuation of one presented to the Society at the New York meeting, May 1, 1926, the author gives a proof of the empirical principle of superposition by which the electric intensities in a layer dielectric for any applied E.M.F. may be calculated when the intensities corresponding to a constant E.M.F. are known. The principle is sometimes stated as valid for the current but this is erroneous. However, the expression follows at once from the expressions for the various electric intensities.

35. Professor R. L. Wilder: *On the definition of simple closed curve.* Second paper.

In an earlier paper of the same title presented to this Society on January 1, 1926, the author gave a definition of a simple closed curve M which required neither that M be bounded nor that it be closed. However, this definition did require that certain subsets of M should be closed in M. The following two definitions do not subject M to any of the above restrictions: (1) A simple closed curve is a connected and regular point set which remains connected upon the omission of any connected subset; (2) A simple closed curve is a connected and regular point set which contains no cut-points and is disconnected by the omission of any two of its points.

36. Dr. H. M. Gehman: *Concerning irreducibly connected sets and irreducible continua.*

Every bounded continuum is irreducibly connected about the set consisting of its non-cut-points. If M is any connected set, which is irreducibly connected about one of its subsets B, then B contains all the non-cut-points of M. Therefore, a necessary and sufficient condition that a bounded continuum M be irreducibly connected about one of its subsets B, is that B contain all the non-cut-points of M. Every bounded continuum is an irreducible continuum about the set consisting of its non-cut-points. If M is any continuum which is an irreducible continuum about a subset consisting of n points, but not about any subset consisting of $n-1$ points, and if A is any closed subset of M such that M is an irreducible continuum about A but not about any proper closed subset of A, then A consists of exactly n points.

37. Dr. H. M. Gehman: *Concerning end points of continuous curves and other continua.*

The property used by K. Menger (MATHEMATISCHE ANNALEN, vol. 95 (1925), p. 277) to define an end point is equivalent in the case of a continuous curve, to the property used by R. L. Wilder (FUNDAMENTA MATHEMATICAE, vol. 7 (1925), p. 358). In the case of a bounded continuum, Menger's property is stronger than Yoneyama's property A, and independent of his properties B and C (TÔHOKU MATHEMATICAL JOURNAL, vol. 18 (1920), p. 253). It is shown that if a point P of a bounded continuum M either has Menger's property, or has Yoneyama's properties

A and C, then M is connected im kleinen at P, and if P is on the boundary of a domain complementary to M, then P is accessible from that domain. Examples are given to show that this theorem does not remain true if the above properties are replaced by certain other weaker properties

38. Mr. G. T. Whyburn: *The most general closed point set over which a continuous function may be defined by certain properties.*

This paper will appear in full in an early issue of this BULLETIN.

39. Mr. G. T. Whyburn: *Concerning certain types of continuous curves.*

This paper establishes the following results: (1) A bounded continuum M is a continuous curve if and only if every connected open subset of M has property S (see R. L. Moore, FUNDAMENTA MATHEMATICAE, vol. 3 (1921), p. 232). (2) A bounded continuum M is an acyclic continuous curve if and only if every connected subset of M is uniformly connected im kleinen. (3) A bounded continuum M is a continuous curve if and only if the set of all non-end points of M is uniformly connected im kleinen. (4) A bounded continuum M is a continuous curve if and only if the set of all non-end points of M has property S. (5) Every strongly connected subset of a continuous curve every subcontinuum of which is a continuous curve is strongly connected im kleinen. (6) The set of all non-end points of a continuous curve M is a subset of the sum of the boundaries of the complementary domains of M provided that if A and B are any two points belonging to the same simple closed curve of M, then $M-(A+B)$ is not connected.

40. Dr. B. O. Koopman: *Note on the extension of a method of Briot and Bouquet for the reduction of singular points.*

This paper appears in full in the present issue of this BULLETIN.

41. Mr. J. H. Sturdivant: *Properties of second-order linear systems with bounded, Lebesgue integrable coefficients.*

Bôcher (*Leçons sur les Méthodes de Sturm*, Paris, 1917) and Ettlinger (TRANSACTIONS OF THIS SOCIETY, vols. 19 and 22) have demonstrated properties of the zeros of a solution of second-order linear systems with coefficients continuous in x and the parameter as independent variables, and of a linear combination of a solution with its first derivative, for both the Sturmian and the general self-adjoint types involving values at two points. By making use of well known properties of Lebesgue integrals and of a forthcoming, extended implicit function theorem due to Ettlinger, this paper extends the above results to systems with coefficients continuous in the parameter, but only bounded and Lebesgue integrable in x.

42. Mr. W. A. Rees: *The method of successive approximations for a second order ordinary linear differential equation with bounded measurable coefficients.*

The method of successive approximation, developed by Peano, Picard and Bôcher, is here applied to ordinary linear differential equations of the second order with coefficients that are bounded and measurable on $a \leqq x \leqq b$. The method follows that of Bôcher (*Leçons sur les Méthodes de Sturm*, Paris, 1917, Chapter I). In addition the existence theorem is established for the system of two ordinary linear differential equations of the first order, $y' = K(x)z + J(x)$; $z' = G(x)y + H(x)$; $y(a) = \alpha$, $z(a) = \beta$, where K, G, H, and J are bounded measurable functions of the real variable x on $a \leqq x \leqq b$, and α and β are real constants. This system is more general than a system of the second order but reduces to the latter if $K(x)$ is continuous and has a bounded measurable derivative $K'(x)$ at every point on $a \leqq x \leqq b$ except for possibly a null set. For this particular case, the existence theorem may be regarded as a second method of obtaining a solution of an ordinary second order system. The errors after n approximations are evaluated and compared for the two methods.

43. Professor H. J. Ettlinger: *Linear boundary value problems with summable coefficients involving integral auxiliary conditions.*

This paper deals with a system of n ordinary linear differential equations of the first order, whose coefficients are summable functions. By means of the adjoint system a necessary and sufficient condition is obtained for the equivalence between the most general type of linear auxiliary integral condition and a linear boundary condition of the usual type involving the values of the unknown functions at the end points. The above result is carried over to an ordinary linear differential equation of the nth order. One of the results is that the self-adjoint pair of equations $y' = K(x)z$, $z' = G(x)y$, where $K(x)$ and $G(x)$ are summable functions of x, with the integral auxiliary conditions $\int_a^b r_i(t)y(t)dt = 0$, $i = 1, 2$, where r_1 and r_2 are independent summable functions, can be carried over into the most general self-adjoint system of the second order involving values of y and z at two points (see TRANSACTIONS OF THIS SOCIETY, vol. 19, p. 79; vol. 22, p. 136).

44. Professor H. J. Ettlinger: *Existence theorems for implicit functions of real variables.*

The classical theorems on implicit functions make use of the continuity of the given functions and their partial derivatives when all the variables are considered as independent. The author wishes to establish the existence of unique and continuous solutions of a system of equations under conditions which demand a great deal less than the above. For a function of the two variables the following theorem is proved: If $F(x, y)$ is de-

fined in the neighborhood of (X, Y); $F(X, Y)=0$; $F(x, Y)$ is continuous in x; $F_y'(x, y)$ exists for every x and for every y almost everywhere, and is bounded; F_y' is measurable in y for every x and continuous in x for every y almost everywhere; $F_y' \geqq 0$ (or $\leqq 0$), the equality sign holding for any x on a null set only, then for some neighborhood of (X, Y) there exists a single-valued continuous function $y=y(x)$ such that $Y=y(X)$ and $F(x, y(x)) \equiv 0$. Additional hypotheses concerning the existence of F_x with properties symmetric to those of F_y', the continuity of F_y' in y for fixed x, the continuity of F_x' in x for fixed y, insure the existence of the derivative $y'(x)$, wherever $F_y' \neq 0$. Corresponding theorems are given for a system of n functions in n dependent variables and m independent variables.

45. Professor H. J. Ettlinger: *R. L. Moore's principle and its converse.*

R. L. Moore (Annals of Mathematics, (2), vol. 13, p. 161) has stated an important principle which the author has made use of in various connections. It is the purpose of this paper to restate this theorem in terms of a sequence of *horizontal functions* defined on a point set E in an m-dimensional closed domain. More general sufficient conditions are obtained which insure the validity of the principle. Necessary conditions are also obtained. Finally, necessary and sufficient conditions are found. The generalization of Moore's principle contained in this paper corresponds to the passage from the bounded measurable case of integrability to the general summable case. Applications are made to obtain properties of summable functions.

46. Professor H. J. Ettlinger: *On the inversion of the order of integration of a two-fold iterated integral.*

Based on the work of W. H. Young, Lichtenstein, and Gillespie, a necessary and sufficient condition is obtained for inverting the order of a two-fold iterated integral for which the double integral does not necessarily exist. More general sufficient conditions than those given by the above writers are also obtained. From the principal theorems, several corollaries follow relative to integrable functions of a parameter defined by a definite integral.

47. Professor H. J. Ettlinger: *The kinetics of learning.*

The laws of certain specified types of learning process, in particular, a mechanical process, which may be roughly represented by learning to operate a typewriter, may be embodied in a first order linear differential (or difference) equation with constant coefficients. From this relation follows a set of equations which express chiefly by means of exponential functions (or binomial expressions) the quantitative aspects of the learning process and exhibit it as a form of "growth." It is not unlikely that equations of this kind may prove useful in analyzing personnel performance, in factory and offices, as well as predicting future attainment.

48. Dr. M. H. Stone: *Integrals analogous to Fourier integrals.*

In this paper the problem of representing an arbitrary function in terms of integrals which generalize the Fourier integral as Sturm-Liouville series do Fourier series, is considered. The basis of the treatment is a passage from a regular differential system of the second order on an interval (a, b) to the limiting cases $a=0, b=+\infty$, and $a=-\infty, b=+\infty$. Hankel's integral in terms of Bessel functions is also discussed by means of formulas suggested by the preceding work. The results may be described roughly by the statement that these integrals are equivalent to the Fourier integral on any interval completely interior to the infinite interval of definition, with respect to the class of all functions absolutely summable on that infinite interval.

49. Professor Marston Morse: *A theory of periodic extremals "in the large."*

In applying the calculus of variations to the study of differential equations, most frequently it is not a minimum or maximum that is sought. Under very general conditions the search for extremals joining two fixed points or deformable into a given closed curve can be reduced to that for the critical points of a function f of a large but fixed number of variables. Of the $n+1$ types of critical points only the two extreme types can give maxima or minima. The nature of the critical point of f corresponding to a closed extremal g is shown to depend in an exact manner upon the number of conjugate points on g, the number of periodic, linearly independent solutions of the corresponding Jacobi differential equation, and upon the angle at which a neighboring extremal returns to intersect itself, if it returns at all. All this is "in the small." The basis for the results "in the large" are the author's *Relations between critical points*, TRANSACTIONS OF THIS SOCIETY, vol. 27 (1925), p. 345.

50. Dr. A. D. Michal: *A generalization of Cartan's integral invariants.*

This paper considers non-additive functional invariants which generalize, and include as a special case, Cartan's integral invariants. The author points out that Cartan's process of stepping from an invariant in Poincaré's sense to a complete Cartan invariant can be extended to certain non-additive functional invariants when the functional invariants in Poincaré's sense are of a particular type. Properties of tensors whose components depend on i points are found useful.

51. Dr. L. S. Hill: *Transformation properties of certain general regions and of their frontiers.*

A general theorem is established which has as a corollary the important theorem that a homeomorphism between two simple closed Jordan curves in the euclidean plane may be extended to the bounded regions determined by the curves. A bounded simply-connected region R in the euclidean

plane is considered, whose frontier F is the sum of two "separable" continua irreducible between two points and is also the frontier of the unbounded component of its complement. It is shown that such a frontier F may be represented as the "supracontinuous" image of the circumference C of a circle; and that the supracontinuous function thus defined over the circumference C may be extended to have the range $C+D$, where D is the interior of the circle, in such manner that the extended function is supracontinuous over $C+D$, and images the set $C+D$ on the set $F+R$. It is further shown that the correspondence thus established between the region R and the interior D of the circle is a homeomorphism.

52. Dr. L. S. Hill: *Properties of certain aggregate functions.*

In Hahn's recent work on the "prime parts" of a continuum, and in papers by others, notably R. L. Moore, the idea has been suggested of treating aggregates of continua as functions with properties of upper semicontinuity, etc. In the present paper, a function $X_t=f(t)$ is studied, the range $T=\{t\}$ of the variable being a set of points in a euclidean space, and the function "value" X_t a set of points in the same, or another, euclidean space. Under suitable restrictions, the properties of such functions are investigated and exhibited in convenient form. Generalizations of theorems in classical real analysis are incidentally obtained. It is shown, for example, that if $T=\{t\}$ is either a region or a perfect set, and the function $X_t=f(t)$ is upper semicontinuous, or lower semicontinuous, at each point of T, then the function is continuous at each point of a set which is everywhere dense in T.

53. Professor I. M. Vinogradov: *On a general theorem concerning the distribution of residues and non-residues of powers.*

In the present paper the author offers a new method for solving certain questions regarding the distribution of residues and non-residues of powers, discussed in his earlier papers of 1916–18. The difference between this method and that previously used lies in its entirely elementary character.

54. Professor W. M. Whyburn: *On a difference system of the second order connected with boundary value problems for differential systems.*

The author treats the system of difference equations

$$z_{in}(\lambda)=K_{in}(\lambda)\cdot\Delta y_{in}(\lambda)/\Delta n,\quad \Delta z_{in}(\lambda)/\Delta n=G_{in}(\lambda)y_{i+1,n}(\lambda),$$

defined over the finite point set $E_n=\{x_{in}\}$ lying on an interval $a\leqq x_i\leqq b$; K_{in} and G_{in} are continuous functions of λ on $L_1<\lambda<L_2$ but their derivatives with respect to λ need not exist. He discusses (1) the separation of the nodes of two linear combinations of the type $\Phi_{in}=\alpha_{in}z_{in}-\beta_{in}y_{in}$ for any fixed value of λ; (2) conditions for non-oscillation of Y and Φ; (3) necessary and sufficient conditions that the zeros of Y move to the left as λ increases; (4) existence and oscillation theorems for Sturmian boundary conditions; (5) the Green's function. He shows, furthermore, that by

means of a sequence of horizontal functions having a measurable function as the limit function almost everywhere on $a \leqq x \leqq b$, the above equations and many of the above results go over as $n \to \infty$ to a differential system defined over a limiting set E, in which the coefficients are measurable functions (see forthcoming papers by the author and by J. H. Sturdivant). Certain of the above results are established for the more general system $\Delta y_{in}/\Delta_n = A_{in}z_{in} + B_{in}y_{in}$, $\Delta z_{in}/\Delta_n = D_{in}z_{in} + E_{in}y_{in}$.

55. Dr. J. M. Thomas: *Note on a differential equation.*

The differential equation $aX' + bX + cY' + dY = 0$, where a, b, c, and d are given functions of the independent variables x and y, X and Y are unknown functions of x and y respectively, and X' and Y' their derivatives, is shown to be equivalent to a system of total differential equations. The equations can therefore be treated by familiar methods.

56. Professor Norman Miller: *On related maxima and minima.*

This paper is concerned with the character of the extremes of n functions of n real variables at a minimizing point, each function being minimized under the condition that the others retain the constant values which they have at that point. Application is made to the elementary symmetric functions and certain sufficient conditions are developed for determining the character of the several extremes of these functions in the case that two of the n variables have equal values.

57. Professor W. Sierpinski: *Nuclear points in the theory of abstract sets.*

This paper appears in full in the present number of this BULLETIN.

58. Professor V. W. Niemytski: *On the third axiom of metric space.*

This article is devoted to the proof that any space which satisfies the following conditions is metrizable. The points of accumulation of the space may be defined in terms of a symmetric, non-negative distance function $\delta(x, y)$ which vanishes if and only if $x = y$, and has the property that if $L\delta(x, y_n) = 0$, $L\delta(y_n, z_n) = 0$, then $L\delta(x, z_n) = 0$. Two proofs are given, one based on an article by A. D. Pitcher and E. W. Chittenden (TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 66), the other on the necessary and sufficient conditions for the metrization of a class (L) formulated by Alexandroff and Urysohn (COMPTES RENDUS, vol. 177, p. 1274).

59. Mr. G. P. Aldrich: *On the complete existential theory of ten properties of the classes (V) of Fréchet.*

This paper presents the results of an investigation of the interrelations of ten fundamental properties of families of neighborhoods. Certain combinations of these properties are found to be completely independent.

Of the 1024 possible combinations of the ten properties and their negatives it is found that only 150 correspond to existing spaces. The 150 examples required to complete the discussion are given. The paper contains the following appendices: On the complete independence of the axioms of Hausdorff for topological space; a postulational definition of betweenness in terms of neighborhood; a definition of point of accumulation in terms of dyadic relations.

60. Miss Fay Farnum: *On triadic Cremona nets.*

A birational correspondence between the points of two planes is called triadic if, in each of the two planes, there are only three groups of fundamental points. Examples of this case have been published by Ruffini and by Montesano. The present paper intends to exhibit further cases of this kind. The author develops (a) triadic correspondences, as products of previously existing solutions; (b) triadic correspondences having simple basis points in both planes; (c) some triadic correspondences having simple basis points in one plane only.

61. Professor J. W. Campbell: *A periodic solution for a certain problem in mechanics.*

If a mass m resting on a smooth horizontal table is connected by a light inextensible string, which passes through a hole in the center of the table, with a mass M hanging freely, then under certain conditions when m revolves about the hole, M will execute periodic motion in a vertical line. A power series solution is obtained for this periodic motion of M when the amplitude is not too large.

62. Professor E. P. Lane: *The contact of a cubic surface with an analytic surface.*

Using Wilczynski's canonical form of the differential equations of a surface referred to its asymptotics, the author investigates the nature of a contact of a cubic surface with an analytic surface, and obtains necessary and sufficient conditions that a surface may be a cubic. The differential equations being written in the form $y_{uu}+2by_v+fy=0$, $y_{vv}+2a'y_u+gy=0$, the conditions that an integrating surface may be a cubic may be reduced to $a'=b$, $f=(3/2)b_v+(1/4)(b_{uu}/b)-(5/16)(b_u/b)^2$, $g=(3/2)b_u+(1/4)(b_{vv}/b)-(5/16)(b_v/b)^2$.

63. Professor H. J. Ettlinger: *On the zeros of functions associated with a linear system of the second order.*

Two lemmas are stated concerning the zeros of continuous functions on an interval: If $f(x)$ is a real continuous function of x on I, $a\leqq x\leqq b$, the set of zeros of $f(x)$ is closed; if $f(x)$ has a derivative at every point of I and if $f^2+f'^2>0$ on I, the set of zeros of $f(x)$ on I is finite in number. By means of these lemmas, the author establishes theorems concerning the number and separation of the zeros of the functions $y(x)$ and $z(x)$,

and linear combinations of these, defined by the differential system $y'=K(x)z(x)$, $z'=G(x)y(x)$, $y(a)=A$, $z(a)=B$ where $K(x)$ is positive and continuous on I and $G(x)$ is bounded and measurable, A and B being real constants not both zero.

64. Miss L. W. Griffiths: *Certain quaternary quadratic forms and diophantine equations.*

A unified treatment of sets of integral elements in the generalized quaternion algebras (see Dickson, *Algebras and their Arithmetics*, p. 187) in which the parameters $(\alpha, -\beta)$ are $(-1, 1)$, $(-2, -3)$, $(-1, -3)$, $(2, -3)$, $(3, -3)$, $(4, -3)$, $(1, 5)$, $(2, 5)$, $(-2, 5)$, $(-3, 5)$, $(1, -7)$, $(-1, -7)$, $(1, 9)$, $(1, -11)$, $(-1, 13)$ yields in application important results concerning the representation of integers in the forms $\sigma^2-\alpha\xi^2+\epsilon\eta^2-\alpha\epsilon\zeta^2+\sigma\eta-\alpha\xi\zeta$, $\sigma^2-\alpha\xi^2-\tau\eta^2+\alpha\tau\zeta^2$, where σ, ξ, η, ζ are integers and $\epsilon=(1-\tau)/4$. The number of such representations is obtained, as are also all solutions of the equations in $k+4$ variables, $k>0$, $\sigma^2-\alpha\xi^2+\epsilon\eta^2-\alpha\epsilon\zeta^2+\sigma\eta-\alpha\xi\zeta=\delta_1\cdots\delta_k$, $\sigma^2-\alpha\xi^2-\tau\eta^2+\alpha\tau\zeta^2=\delta_1\cdots\delta_k$. Certain definite exceptions appear with $(\alpha, \tau)=(3, -3)$, $(4, -3)$, $(-1, -7)$. The method is essentially that of Dickson, (American Journal of Mathematics, vol. 46 (1924)).

65. Professor G. C. Evans: *Note on a theorem of Noaillon and the Dirichlet problem for the circle.*

A necessary and sufficient condition that $u(r, \theta)$ be given by the Poisson integral in terms of some summable boundary function $f(\theta)$ has been given by the author, (Comptes Rendus, vol. 177 (1923), p. 241), in terms of uniform absolute continuity of the integral $\int u(r, \theta)d\theta$, where $u(r, \theta)$ is harmonic within the circle; also by Noaillon in terms of convergence in the mean to the function $f(\theta)$ (Comptes Rendus, vol. 182 (1926), p. 1371). The latter condition may easily be shown, for harmonic functions, to be equivalent to the former, but even in a slightly amended form it does not involve so essentially the properties of harmonic functions as the former. There is still room indeed for a necessary and sufficient condition which will partake less of the general theory of limits of integrals than any of those so far given but will depend more specifically upon the properties of a function harmonic inside the circle.

66. Professor J. L. Walsh: *On the degree of approximation to an analytic function.*

Let C be an arbitrary limited point set in the z-plane, whose complementary set is a simply connected region B (therefore open). Let $u=\phi(z)$ be a function which maps B on the exterior of the unit circle in the u-plane, so that the infinite points correspond to each other. Let C_R $(R>1)$ denote the Jordan curve $|\phi(z)|=R$, that is, the transform in the z-plane of the circle $|u|=R$. A necessary and sufficient condition that an arbitrary function $F(z)$ which is defined on C be regular-analytic

on C is that polynomials $V_n(z)$ of degree n, for $n=0, 1, 2, \cdots$, exist so that we have (1): $|F(z)-V_n(z)| \leqq (M/R^n)$ (M, $R>1$, constant), for every z on C. When the polynomials $V_n(z)$ are given, so that (1) holds, $F(z)$ is regular-analytic in the interior of C_R. When $F(z)$ is regular-analytic in the closed interior of $C_{R'}$, then (1) holds, if the polynomials are suitably chosen, for $R=R'$.

67. Professor Dunham Jackson: *Note on a tensor of the second rank in function space.*

This paper appears in full in the present issue of this BULLETIN.

68. Professor C. E. Wilder: *Reduction of the ordinary linear differential equation of the nth order whose coefficients are certain polynomials in a parameter to a system of n first order equations which are linear in the parameter.*

In this paper the following theorem is proved: The differential equation $(d^n y/dx^n)+P_{n-1}(x, \rho)(d^{n-1}y/dx^{n-1})+\cdots+P_0(x, \rho)=0$, in which $P_k(x, \rho)=\sum_{j=0}^{n-k} P_{n-k,j}(x)\rho^j$, $k=0, 1, 2, \cdots, n-1$, may be reduced to a system of n linear differential equations of the form $(dy_i/dx)=\sum_{j=1}^{n}[\rho a_{ij}(x)+b_{ij}(x)]y_j$, $i=1, 2, \cdots, n$, in which the functions $a_{ij}(x)$ and $b_{ij}(x)$ are continuous in a given interval $a \leqq x \leqq b$ if the functions $P_{i+k,i}(x)$ possess $n-k$ continuous derivatives in this interval and if the "characteristic equation" $\sum_{t=0}^{n} P_{n-t,n-t}(x)a^t=0$ has roots which are distinct for all values of x in the interval.

69. Professor J. D. Tamarkin: *On Fredholm's integral equations whose kernels are analytic in a parameter.*

This paper is concerned with Fredholm's integral equation whose kernel $K(x, \xi, \lambda)$ is analytic or meromorphic in λ. By using infinite determinants, it is established that the reciprocal is also meromorphic in λ, under certain assumptions as to the kernel $K(x, \xi, \lambda)$. In the special case of the classical Fredholm integral equation a representation is obtained in the form of a quotient of two infinite determinants, which are entire functions in λ, under more general conditions than it has been done before.

70. Professor G. E. Raynor: *On the generalization of Beltrami equations to n-space.*

In the TRANSACTIONS OF THIS SOCIETY for October, 1925, Hedrick and Ingold have obtained a generalization of the Beltrami equations to 3-space. In the present paper the writer obtains equations of the same sort for n-space by a different method and of a somewhat more explicit form.

ARNOLD DRESDEN,
Assistant Secretary.

RECENT PROGRESS WITH THE DIRICHLET PROBLEM*

BY O. D. KELLOGG

1. *The Classical Problem of Dirichlet, and its Status at the Beginning of the Twentieth Century.* Let T denote an open continuum in a euclidean space of n dimensions, and let t denote the set of its boundary points. A function, $F(p)$, defined on t, will be said to be continuous, if to every point, p, of t, and every $\epsilon>0$, there corresponds a $\delta>0$, such that $|F(p)-F(q)|<\epsilon$ for every point, q, of t, for which the distance $pq<\delta$. The classical problem of Dirichlet is then the following. *Given T and $F(p)$, to find a function continuous in $T+t$, reducing to $F(p)$ on t, and having in T continuous partial derivatives of second order which satisfy Laplace's differential equation*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

We shall limit ourselves to the cases $n \leqq 3$.

We shall be concerned rather with the existence of the solution than with its actual construction. A domain, T, for which a solution exists corresponding to each continuous $F(p)$, will be called a *normal* domain.

In one dimension, the problem is always possible, for it amounts merely to finding a straight line through two points. In two and three dimensions, the most extensive results at the close of the last century were due to Poincaré and his method of *balayage*.† The contribution of Poin-

* An address delivered at the meeting of the Society on January 2, 1926, in New York City, by invitation of the Program Committee.

† *Sur les équations aux dérivées partielles de la physique mathématique*, AMERICAN JOURNAL, vol. 12 (1890), pp. 211–294, and *Théorie du Potentiel Newtonien*, Paris, 1899. Carried out in two dimensions by A. Paraf, *Sur le problème de Dirichlet*, ANNALES DE TOULOUSE, vol. 6 (1892), pp. H. 1–H. 75.

caré consisted in three elements: first, a recognition of the fact that if the problem was solvable when $F(p)$ represented the boundary values of any polynomial, it was solvable for any continuous $F(p)$,* secondly, a method for establishing a harmonic function (solution of Laplace's equation) corresponding to given boundary values, $F(p)$, and thirdly, a method of proof that this harmonic function assumed the assigned boundary values by means of certain dominant functions which we shall call, following Lebesgue, *barriers*, and consider later (§5). The result was that the existence of the solution of the Dirichlet problem was established for domains bounded by surfaces with finite principal curvatures at all but a finite number of ordinary conical points.

The method of integral equations, developed by Fredholm,† because of its powerful character, gave great impetus to the study of the Dirichlet problem. It emphasized the independence of the existence theorem of the general shape of the boundary of T, and proved most useful in many problems of potential theory. But it was hindered in aiding substantially the progress of the Dirichlet problem by its use of the double distribution, which carried with it the demand for a fairly smooth boundary.

The solution of a certain fundamental case of the Dirichlet problem was brought to a successful conclusion by Osgood‡ in his proof that Green's function existed for the most general simply connected domain of the plane with more

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* The reasoning was not complete, for it involved the assumption that corresponding to continuous $F(p)$ there always existed a function $F(P)$, continuous throughout $T+t$ and coinciding with $F(p)$ on t, a fact which had not been established at that time.

† *Sur une nouvelle méthode pour la résolution du problème de Dirichlet*, OFVERSIGT AF KONGL. VET. AK. FÖRH., vol. 57 (1900), pp. 39 ff. *Sur une classe d'équations fonctionnelles*, ACTA MATHEMATICA, vol. 27 (1903), pp. 365–390.

‡ *On the existence of the Green's function for the most general simply connected plane region*, TRANSACTIONS OF THIS SOCIETY, vol. 1 (1900), pp. 310–314.

than one boundary point. That this result implied the fact that such a domain was normal was recognized only sixteen years later, when Lichtenstein* proved that from the existence of Green's function for a domain, T, followed the possibility of the Dirichlet problem for any continuous $F(p)$ for that domain.

But in the mean time, Lebesgue, using the methods of the calculus of variations as refounded by Hilbert,† had attained results including this.‡ He showed that a domain T in two dimensions was normal provided no boundary point could be enclosed by a curve lying in T and an arbitrarily small neighborhood of the point. In three dimensions, Zaremba§ proved that T was normal, provided that each boundary point was the vertex of a right circular cone lying outside of T except at the vertex.

2. *Limitations on the Dirichlet Problem.* Thus far, writers on the subject had regarded the Dirichlet problem as always possible, and their endeavors had been to find methods of proof powerful enough to establish its possibility. A new chapter begins with the recognition of the fact that domains exist, for $n \geqq 2$, which are not normal. In 1903, Bôcher|| had proved that a singularity of a bounded harmonic function at an isolated point can only be a removable one. Zaremba¶ pointed out that this meant that a domain whose boundary contained an isolated point could not be normal. An even more striking example was given for a

* *Ueber die erste Randwertaufgabe der Potentialtheorie*, SITZUNGSBERICHTE, BERLINER MATHEMATISCHE GESELLSCHAFT; ARCHIV FÜR MATHEMATIK UND PHYSIK (3), vol. 25 (1916–17), pp. 92–96.

† *Über das Dirichletsche Prinzip*, JAHRESBERICHT DER VEREINIGUNG, vol. 8 (1900), p. 184 et seq.

‡ *Sur le problème de Dirichlet*, RENDICONTI DI PALERMO, vol. 24 (1907), pp. 371–402.

§ *Sur le principe de Dirichlet*, ACTA MATHEMATICA, vol. 34 (1911), pp. 293–316.

|| *Singular points of functions which satisfy partial differential equations of the elliptic type*, this BULLETIN, vol. 9 (1903), pp. 455–465.

¶ ACTA MATHEMATICA, vol. 34, p. 310.

three-dimensional domain by Lebesgue*; the domain was simply connected, but its boundary contained a sharp inward projecting spine, and a certain harmonic function assuming preassigned continuous boundary values everywhere else failed to assume the given value at the point of the spine. With the emergence of facts of this character, the study of the Dirichlet problem was forced into new channels, and later researches may be fairly characterized as falling under two heads: the recognition and characterization of exceptional boundary points, and the suitable extension of the meaning of the Dirichlet problem, so that it shall admit of one, and just one, solution, no matter what the domain.

3. *The Extended Function of Green.* Given a domain T, there exists a sequence of domains $[T_n]$ with these properties: each is a part of all the succeeding ones, and each point of T is in all of the domains of the sequence from a given one on. For instance, T_n might be the interior points of the set of all the closed cubes of the lattice $x=a/2^n$, $y=b/2^n$, $z=c/2^n$, (a, b, and c being integers) which lay in T. Such domains would be normal, as shown by Zaremba. If desired, these domains could also be so rounded off as to have continuous principal curvatures at all points. Also, it could be arranged that the boundary of each was also interior to those following. We shall speak of such a sequence as a *set of nested domains with T as limit*, not necessarily requiring, however, that they be rounded off. We shall always suppose them normal.

Then, T being a domain, and Q any point of T, Green's function $G_n(P, Q)$ exists for each domain T_n of a nested set as soon as T_n contains Q. The sequence $[G_n(P, Q)]$ is monotone increasing, and, by the reasoning of Harnack,† approaches a limit, uniformly in any closed sub-region of T

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* *Sur des cas d'impossibilité du problème de Dirichlet*, COMPTES RENDUS DE LA SOCIÉTÉ DE FRANCE, 1913, p. 17 et seq. See 6, below.

† *Grundlagen der Theorie des logarithmischen Potentials*, Leipzig, 1887, §39.

which omits Q. This limit function, $G(P, Q)$, is harmonic in the sub-region. We call it Green's function for T, leaving open the question as to its behavior on the boundary, t.* It has these properties: $G(P, Q)-1/PQ$, is harmonic in T save for a removable singularity at Q; this difference is never positive; if T extends to infinity, $G(P, Q)$ vanishes regularly at infinity; and $G(P, Q)=G(Q, P)$.

The points p of t at which $G(P, Q)$ approaches 0 are called *regular* boundary points. All other points of t are called *exceptional*. If there are no exceptional boundary points, T is normal.† But if there are exceptional points, the domain is not normal, as may be seen as follows. The Dirichlet problem for boundary values $1/PQ$ is not possible. For, if it were, there would exist a function of Green approaching 0 at every boundary point. Such a function would dominate all the $G_n(P, Q)$ of our sequence, and hence their limit, $G(P, Q)$ The Green function formed in this way would then also approach 0 at every boundary point, and exceptional boundary points would then not exist.

The above definition of regular and exceptional boundary points is easily shown to be independent of the pole, Q, of Green's function. In fact, if Q and Q' be two points of T, and R is a closed region containing them in its interior, a certain constant multiple of $G(P, Q')$ dominates $G(P, Q)$ in $T-R$, so that $G(P, Q)$ approaches 0 at every boundary point at which $G(P, Q')$ does, and vice versa.‡

4. *Wiener's Sequence Solution of the Dirichlet Problem.* Given a domain T, and a function, $F(p)$, continuous on its boundary, t, it is always possible to find a function $F(P)$, defined and continuous throughout $T+t$, and coinciding

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* Kellogg, *On the general solution of the classical Dirichlet problem*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 12 (1926), pp. 397–406. Referred to later as I.

† Lichtenstein, loc. cit. In three dimensions, Kellogg, loc. cit., I.

‡ Kellogg, loc. cit., I.

with $F(p)$ on t.* This fact permits the establishment of a function, harmonic in t, and corresponding to any continuous boundary values $F(p)$, in the same manner as that in which Harnack established Green's function.† Thus, given T and the continuous boundary values $F(p)$, we determine any continuous extension, $F(P)$, of $F(p)$ to the points of T, and any nested set $[T_n]$ of normal domains. For each of these, we determine the function $u_n(P)$, harmonic in T_n, and approaching on the boundary, t_n, the values which $F(P)$ there assumes. If T is normal, the sequence $[u_n(P)]$ converges uniformly to a function, harmonic in T, which assumes the boundary values $F(p)$.‡ It is the great merit of Wiener to have established the facts that this sequence always converges, no matter what the domain T or the continuous boundary values, $F(p)$, and that its limit, $u(P)$, is harmonic in T, and is independent of both the continuous extension of $F(p)$ to the points of T, and of the nested set $[T_n]$.§ This function $u(P)$ also approaches the boundary values $F(p)$ at all regular boundary points. It is thus, in an extended sense, a solution of the Dirichlet problem which exists no matter what the domain T or the continuous boundary values $F(p)$. We call it the *sequence solution*. We shall see later (§9) in what sense it may as yet be regarded as a final solution of the Dirichlet problem.

5. *The Character of Boundary Points. Barriers.* The method of balayage, the methods of the calculus of variations, and the above extended method of Harnack, all produce harmonic functions as limits of sequences corresponding to assigned boundary values, and all require sup-

* Lebesgue, RENDICONTI DI PALERMO, vol. 24, p. 379; Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1918, pp. 617–20.

† Kellogg, *An example in potential theory*, PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 58 (1923), pp. 527–533. Referred to later as II.

‡ Ibid.

§ *Certain notions in potential theory*, JOURNAL OF MATHEMATICS AND PHYSICS, Mass. Inst. of Tech., vol. 3 (1924), pp. 24–51. A simplified treatment of these theorems will be found in the writer's paper I, cited above.

plementary investigation of the behavior of this harmonic function on the boundary.* It was Lebesgue who first explicitly separated the two problems, and it was Lebesgue† who first generalized the barrier concept so effectively used by Poincaré. We shall define barrier in its simplest aspect; essentially the generality of other definitions is only apparent, as we shall see. Let T be a bounded domain, and p one of its boundary points. The function $U(P, p)$ is a *barrier* for T at p if it is harmonic in T, never less than Pp, and if it approaches 0 at p. Using this concept, we may state the theorem: A necessary and sufficient condition that T be normal is that there exist a barrier for T at every boundary point of T.‡

We can go farther than this, however, and find in the barrier a criterion for the regularity of the individual boundary points. But it is first desirable to consider the notions of *subharmonic* and *superharmonic* functions. According to F. Riesz,§ a function $W(P)$, defined and continuous in T (or upper semicontinuous—and it may even become negatively infinite at isolated points), is subharmonic when it possesses the following property: given any subdomain, T', of T, every function which is harmonic in T' and continuous in $T'+t'$ and which dominates $W(p)$ on t', does so also throughout T'. Here t' denotes the boundary of T'. A superharmonic function may be defined simply as the negative of a subharmonic function. A harmonic function belongs to both classes. The following are a few of the properties of subharmonic functions given by Riesz, and which are of importance for our present purposes. Let $A_\sigma W$ denote the arithmetic mean of the values of W

* Loc. cit., RENDICONTI DI PALERMO, vol. 24.

† *Sur le problème de Dirichlet*, COMPTES RENDUS, vol. 154 (1912), p. 335.

‡ Kellogg, loc. cit., II.

§ *Ueber subharmonische Funktionen und ihre Rolle in der Funktionentheorie und in der Potentialtheorie*, ACTA, FRANZ-JOSEPH UNIVERSITY, SZEGED, vol. 2 (1925), pp. 87–100.

on the sphere σ with center at C, and lying in a domain in which W is continuous. Then W is subharmonic in this domain if for every such sphere, $W(C) \leqq A_\sigma W$, and conversely. If W has continuous second partial derivatives, $\nabla^2 W \geqq 0$ is a necessary and sufficient condition that W be subharmonic. Finally, if $[W]$ denote a set (which, for our purposes, may be finite) of functions subharmonic in T, and if $W(P)$ has at each point P of T the greatest of the values which the functions of the set assume at P, then $W(P)$ is also subharmonic.

It is now easy to show that a necessary and sufficient condition that p be a regular boundary point of T is that a barrier exist for T at p. We first consider the case in which T is bounded. Let us assume that a barrier $U(P, p)$ exists. Then $G(P, Q) \to 0$ as $P \to p$, and p is regular. For if σ be a sphere, of radius a about Q, lying in T, and if m be the minimum distance of the points of σ from t, then on σ,

$$\frac{1}{ma} U(P, p) \geqq \frac{1}{ma} Pp \geqq \frac{1}{a} = \frac{1}{PQ}.$$

Hence in $T_n - \sigma$,

$$G_n(P, Q) \leqq \frac{1}{ma} U(P, p),$$

and hence in $T - \sigma$,

$$G(P, Q) \leqq \frac{1}{ma} U(P, p),$$

so that $G(P, Q) \to 0$ as $P \to p$. On the other hand, if p is regular, a barrier $U(P, p)$ exists. It is the sequence solution of the Dirichlet problem corresponding to the boundary values which $F(P) = Pp$ assumes. For if we denote Pp by r, $\nabla^2 F(P) = \nabla^2 r = 2/r > 0$, and $F(P)$ is subharmonic in T. Hence the sequence functions, $u_n(P)$, dominate $F(P)$, and so, therefore, does their limit, $U(P, p)$. But p is a regular point, and hence at p the sequence solution $U(P, p)$ approaches the assigned boundary value, 0.

So much for bounded domains. A simple expedient shows that if a barrier $U(P, p)$ exists for only so much of T as is contained in a sphere, σ, about p, p is regular, and conversely. Let a denote the radius of σ. We define a continuous superharmonic function $W(P, p)$ as follows: within σ, $W(P, p)$ is at P the smaller of the numbers a and $U(P, p)$, or is their common value if equal; in the rest of T, $W(P, p) = a$. Thus the existence of a barrier at p for merely *a neighborhood* of p in T implies the existence of the function $W(P, p)$ which function is superharmonic in T, not less than Pp in a neighborhood of p, and approaches 0 at p. A slight modification of the previous reasoning shows that $G(P, Q)$ must approach 0 at p. On the other hand, if p is a regular boundary point of T, it is also a regular boundary point of any sub-domain T' of which it is a boundary point, since Green's function for T dominates that for T'. Hence $U(P, p)$ exists at p for a neighborhood of p in T, and the superharmonic function $W(P, p)$ exists in T.

Incidentally, the fact has emerged that the regularity of a boundary point is dependent purely on the *im Kleinen* character of the boundary at that point.*

The function $W(P, p)$ is a special case of the more general type of barrier defined by Lebesgue.† While the existence of either implies that of the other, the latter type is often more convenient to apply. $W(P, p)$ is said to be a barrier for T at p if $W(P, p)$ is continuous and superharmonic in T and if it approaches 0 at p and has a positive lower bound at every other boundary point. If p is regular, such a function exists in the more special function established above. On the other hand, if $W(P, p)$ exists, the more special barrier $U(P, p)$ exists for as much of T as lies in

* Lebesgue, *Sur le principe de Dirichlet*, COMPTES RENDUS, vol. 155 (1912), pp. 699–701. Kellogg, loc. cit., II.

† *Conditions de régularité, conditions d'irrégularité, conditions d'impossibilité dans le problème de Dirichlet*, COMPTES RENDUS, vol. 178 (1924), p. 349.

a sphere of radius R about p. For, outside a sphere σ' of radius a about p, and in the part T' of T within the sphere of radius R, $W(P, p)$ has a positive lower bound, δ. This follows for boundary points, by definition, and for interior points because a superharmonic function can have no interior minima, since for them $W(C) \geqq A_\sigma W$. Hence, in $T' - \sigma'$,

$$W(P,p) \geqq \frac{Pp - a}{R - a}\delta,$$

so that

$$W'(P,p) = \frac{R-a}{\delta} W(P,p) + a$$

dominates Pp in T''. But this function is continuous and superharmonic in T', and hence dominates the sequence functions $u_n(P)$ corresponding to $F(P) = Pp$, and therefore their limit, $U(P, p)$. But $W'(P, p) \to a$ as $P \to p$. Hence the upper limit of $U(P, p)$ at p is not greater than a. But it is independent of a, which number may be any small positive quantity, and hence this upper limit is 0. That is, $U(P, p)$ is a barrier for T' in the narrower sense.

6. *Types of Regular and Exceptional Boundary Points.* In the plane, a simply connected region with more than one boundary point has no exceptional points, as we have seen (p. 603). It follows from the *im Kleinen* character of regularity that any boundary point is regular if it belongs to a connected set of boundary points, containing other points. Hence if a plane domain is to have exceptional points, either its boundary must contain isolated points, or the domain must have infinite connectivity. As instances of possibilities, we give the following two examples. The first exhibits a domain of infinite connectivity whose boundary is a linear set of Borel measure 0, yet all of whose points are regular.* In fact, the boundary set in question is Cantor's nowhere dense perfect set, being the points left of a closed segment after the removal of its open middle

* Kellogg, loc. cit., II.

third, the removal of the open middle thirds of the two remaining segments, the removal of the middle thirds of the four remaining segments, and so on, ad infinitum.

The second example* is, in a way, an antithesis to the first. Here a boundary point is exceptional although it is a point of condensation of the boundary. One forms first the function

$$U(x,y) = \sum_{0}^{\infty} \frac{1}{2^n} \log\left[(x - 1/2^n)^2 + y^2\right]^{-1/2},$$

harmonic except at the points $(1/2^n, 0)$. It is not difficult to show that if (x, y) is any regular point of $U(x, y)$,

$$\lim_{n\to\infty} U\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 2 \log 2,$$

and that the locus $U(x, y) = 5$ consists of closed oval curves each enclosing one and but one of the points $(1/2^n, 0)$. The domain, T, in question, consists of the points interior to a simple closed equipotential, $U(x, y) = -C$ (C a sufficiently large positive constant), except for the ovals $U(x, y) = 5$, their interiors, and their single new limit point, O, the origin. A function harmonic in T and approaching the boundary values $-C$ on the outer boundary, and 5 on the ovals and at O, does not exist. For, if it did, it would differ from $U(x, y)$ by a function harmonic in T, bounded, and approaching the boundary values 0 at every point but O. The difference would then vanish, being dominated (as well as its negative) by $-\epsilon \log[(x^2+y^2)/R^2]$ for every positive ϵ, where R is the maximum diameter of T. But $U(x, y)$ does not assume the required boundary value at O. Hence the Dirichlet problem in question has no solution, and T is not normal. But as all other boundary points are regular, O must be exceptional.

Thus points of condensation of the boundary may be exceptional. But a point cannot be regular unless it is a point of condensation of the boundary (see p. 620).

* The underlying idea in the construction of this example, I owe to a conversation with Professor Wiener. A somewhat similar example is given by Lebesgue, loc. cit., RENDICONTI DI PALERMO, vol. 24.

An isolated boundary point, and the point O of the above example, are exceptional points of essentially different characters. The isolated point can be the seat of no singularities of bounded harmonic functions other than removable ones, and it therefore has no influence on the determination of a harmonic function by its boundary values. Such points may therefore be removed from the boundary without essential alteration of the Dirichlet problem. On the other hand, the point O of the example given, cannot be removed without impairing the essential quality of the domain T of being an open set, or else removing regular points at the same time. Bouligand* has made the distinction between the two kinds of exceptional boundary points on the basis of the generalized Green function. A set of points at which $\lim G(P, Q) > 0$,(and such are all isolated points), he calls an *improper* set. At each point of an improper set, not only is $\underline{\lim}\, G(P, Q) > 0$, but $\lim G(P, Q)$ exists and is positive.†

In a space of three dimensions, arcs of regular curves of finite length which are isolated from the rest of the boundary, contain only exceptional points—in fact, they are improper sets.‡ The first example of an exceptional point which is not of the improper type was given by Lebesgue,§ and it is to be noted that in distinction to the possibilities in the plane, this point belongs to the boundary of a simply connected domain. Consider the potential of a spread on a straight line segment whose density varies with the distance from one end:

$$V = \int_0^1 \frac{\xi\, d\xi}{PQ}, \qquad PQ = ((\xi - x)^2 + y^2 + z^2)^{1/2}.$$

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* *Sur le problème de Dirichlet*, Annales de la Société Polonaise, 1925, pp. 59–112.

† Kellogg, loc. cit., I.

‡ Lebesgue, *Sur les singularités des fonctions harmoniques*, Comptes Rendus, vol. 176 (1923), pp. 1097–99.

§ Loc. cit., p. 604, footnote 1.

The integral may be evaluated, and gives

$$V = (1 - 2x + r^2)^{1/2} - r + x \log \frac{(1 - 2x + r^2)^{1/2} + 1 - x}{r - x},$$

where

$$r = (x^2 + y^2 + z^2)^{1/2}.$$

This relation is equivalent to

$$r = x + e^{\mu}, \qquad \mu = \frac{1 - V + \eta}{2x},$$

where η vanishes with r. For a meridian curve of the equipotential surface $V = \text{const.}$, we find

$$y = \pm e^{\mu} (2x + e^{\mu})^{1/2}.$$

These meridian curves, for $V > 1$, enclose the loaded segment, except near the origin, 0, where they have contact with it of infinite order.

The equipotential surface $V = 2$, together with the point 0, bounds an infinite domain, T. The Dirichlet problem for T and the boundary value 2 is impossible. The function V, which fulfills every other requirement, fails to approach 2 at 0. In fact at this point V approaches 1 along every ray terminating at 0 other than the loaded segment.

Lebesgue* has inferred from this example that p is exceptional for a domain T if all the boundary near p lies inside a surface of revolution whose meridian curve is of the form $y = e^{-A/x}$, $(x > 0, A > 0)$, p being at the vertex.

On the other hand, by means of superharmonic barriers of the form $U = (x^{\alpha} + y^{\alpha} - z)^{1/2}$, $\alpha = 2/(2n+1)$, he shows that p is regular if it is the vertex of a surface of revolution of the form $z = Ar^{\beta}$, $(A > 0,\ 0 < \beta < 1,\ r = (x^2 + y^2)^{1/2})$, which lies outside of T in a neighborhood of p, except at p itself. The results are extended to the case of certain spines which are not surfaces of revolution.

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* Comptes Rendus, vol. 178 (1924), p. 349.

Bouligand* has constructed barriers for regions with a general type of reentrant conical point as follows. $U = r^\alpha \psi(\phi, \theta)$ is harmonic in the interior of any cone in which $\psi(\phi, \theta)$ is a continuous differentiable solution of the equation $\Delta_2\psi + \alpha(\alpha+1)\psi = 0$, where $\Delta_2\psi$ is the second differential parameter of Beltrami for the surface of the unit sphere. Let S denote a simply connected domain of this sphere with more than one boundary point. Green's function, $g(P, Q)$, exists for S and the differential equation $\Delta_2 g = 0$, and any solution of the differential equation for ψ which vanishes everywhere on the boundary of S satisfies the integral equation

$$\psi(P) = \frac{\alpha(\alpha+1)}{2\pi} \int\int_S g(P,Q)\psi(Q)dS_Q,$$

and conversely. There exists a solution $\psi_0(\phi, \theta)$, of this equation, positive in S, approaching 0 on the boundary, and corresponding to a positive characteristic number, λ_0. Accordingly, if α is the positive root of $\alpha(\alpha+1) = 2\pi\lambda_0$, $U(P) = r^\alpha \psi_0(\phi, \theta)$ is a barrier for the origin for any domain which admits as strictly exterior (save at the origin), as much of the cone subtended at the origin by the complement on the unit sphere of S as lies in some neighborhood of the origin. For U will be harmonic in T in a neighborhood of the origin, and will approach 0 there but at no other boundary point of T in this neighborhood.

In particular, the domain S may consist of the whole surface of the unit sphere save for an arc of a great circle, or other curve, so that p is regular for T if it is the vertex of a triangle, either flat, or conical and bounded by two elements and a third geodesic, which lies outside T except at p.

Raynor† obtained the following condition. The point p is regular for T provided there is a number $\eta > 0$, such

* *Application de la notion du prolongement des fonctionnelles à l'étude de l'existence de la solution du problème de Dirichlet harmonique*, BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 48 (1924), pp. 1–19.

† *Dirichlet's problem*, ANNALS OF MATHEMATICS, (2), vol. 23 (1923), pp. 183–197.

that on an infinite number of spheres about p with radii approaching 0, the ratio of the Lebesgue measure of the set of points of each sphere exterior to T to the whole area of that sphere, is greater than η.

Bouligand* has given special attention to the character of the infinite point as a boundary point. It is not difficult to show that an inversion which carries a finite boundary point into a finite point does not alter the regularity or exceptional character of this boundary point. Accordingly, it is natural to define the infinite point as regular or exceptional according to the character of the point into which it is transformed by an inversion. If p is a finite exceptional boundary point for a domain T, there exists a sequence of points of T, $[P_k]$ with p as limit, such that $G(P_k, Q) \to \eta > 0$, as $P \to p$. Accordingly, $u_p(P) = \lim G(P, P_k)$ is a never negative, not identically vanishing function, harmonic in T, approaching 0 at all regular boundary points, and dominated by $1/Pp$. An inversion in p leads to this result: a necessary condition that the infinite point be exceptional for a domain T' is the existence of a function positive, bounded, and harmonic in T' and vanishing at all finite regular boundary points. The converse can also be proved, provided the exceptional points of T' are not too numerous. From this Bouligand also inferred independently the result of Raynor.

The above particular criteria for the character of boundary points have been given in some detail to illustrate methods of investigation. It should be added, however, that Wiener† has shown how they may be derived from a general criterion of his in terms of the capacity of point sets, a topic to which we devote the next section. We close the present section with a remarkable result of Bouligand's:‡ If p is an ex-

* *Sur les principes de la théorie du potentiel*, BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 48 (1924), pp. 1–12.

† *The Dirichlet problem*, JOURNAL OF MATHEMATICS AND PHYSICS, Mass. Inst. Tech., vol. 3 (1924), pp. 127–146.

‡ Loc. cit., ANNALES DE LA SOCIÉTÉ POLONAISE.

ceptional point of T, and if $U(P)$ is the sequence solution of the Dirichlet problem corresponding to boundary values which are 0 in the neighborhood of p, and nowhere negative, then the arithmetic mean of the values of $U(P)$ on a sphere about p approaches, as the radius approaches 0, the upper limit of $U(P)$ at p. As the limit of the ratio of the measure of the set of points outside T and on the sphere to the whole area is 0 as the radius approaches 0, the fact that $U(P)$ may not be defined at all points of the spheres causes no difficulty in interpreting this arithmetic mean.

7. *Capacity.* The electrostatic capacity of a conductor may be defined as the charge necessary to raise it to unit potential, when isolated. The concept may be applied to any bounded point set (and we need it for no others) as follows. Let B be such a set, and B' the set obtained by adjoining to B its limit points. Now B' may bound various domains, but B' will certainly contain the complete boundary t, of a domain T, extending to infinity. The sequence solution of the Dirichlet problem for T and the boundary values 1 on t (and vanishing at infinity) is called the *conductor potential* of the set B. If $v(P)$ denote this conductor potential, Gauss' integral,

$$\frac{1}{4\pi}\iint \frac{\partial v}{\partial n}\,dS,$$

taken over any smooth simple surface containing B in its interior, gives the *capacity* of B. A number of properties of the capacities of sets have been derived by Wiener, Bouligand, and the author.* But the outstanding service which the concept has rendered lies in Wiener's necessary and sufficient condition for the regularity of a boundary point: Let p denote a boundary point of T, and let γ_n denote the capacity of the set of points not in T whose distances from p lie in the closed interval $(\lambda^{n+1}, \lambda^n)$, where λ is a number between 0 and 1. Then p is re-

* Loc. cit., I.

gular or exceptional according as the series $\sum_1^\infty (\gamma_n/\lambda^n)$ diverges or converges.

Wiener has given the corresponding theorem for 2 and $n>3$ dimensions.

A proof of the above theorem, somewhat modified, and free from the use of Stieltjes integrals, follows. We first assume the series $\sum_1^\infty (\gamma_n/\lambda^n)$ divergent, and show how it follows that a barrier for T at p can be constructed. From our assumptions, it follows that one of the three series $\sum_1^\infty (\gamma_{3n}/\lambda^{3n})$, $\sum_1^\infty (\gamma_{3n+1}/\lambda^{3n+1})$, $\sum_1^\infty (\gamma_{3n+2}/\lambda^{3n+2})$, must diverge. To fix ideas, suppose it is the first—the notation, not the reasoning, would have to be altered to meet the other two cases. Let B_n denote the point set of the theorem, whose capacity is γ_n, and let $V_m(P)$ denote the conductor potential of the set containing all the points of B_{3m}, B_{3m+3}, B_{3m+6}, $\cdots$. The heart of the argument is to show that $V_m(P)\to 1$ as $P\to p$.

This is done by building up a sequence of harmonic functions which are dominated by $V_m(P)$. The first is harmonic in the region within the sphere σ_{3m-1}, with center at p and radius λ^{3m-1}, except for the set B_{3m} (and also except for any domains which B_{3m} may completely bound, if there are any). It is the sequence solution, $v_1(P)$, corresponding to values 0 on σ_{3m-1} and 1 on B_{3m}. Then we have $V_m(P)\geqq v_1(P)$ at the points at which both are harmonic. We seek a lower bound for $v_1(P)$ on σ_{3m+2}.

By definition, $v_1(P)$ is the limit of a sequence of functions $[u_n(P)]$, $u_n(P)$ being harmonic in a normal region T_n, bounded by σ_{3m-1} and by a smooth surface, t_n, interior to σ_{3m-1} and enclosing B_{3m}, the domains T_n having the domain of $v_1(P)$ as limit, and the boundary values of $u_n(P)$ being 0 on σ_{3m-1} and 1 elsewhere. Then

$$u_n(P) = \frac{1}{4\pi}\int\int\left(G\frac{\partial u_n}{\partial n} - u_n\frac{\partial G}{\partial n}\right)dS_Q,$$

where G is Green's function for σ_{3m-1}, and the integral is taken over the boundary of T_n; or, since u_n and G vanish

on σ_{3m-1}, over t_n. The positive sense of the normal n is the outward one. Accordingly,

$$u_n(P) = \frac{\overline{G}}{4\pi} \iint \frac{\partial u_n}{\partial n} dS_Q,$$

where $\overline{G}$ is a mean value of G. Now $\partial u_n/\partial n$ is never less than the corresponding derivative of the conductor potential of t_n, and hence* than that of B_{3m}, and therefore $u_n(P) \geqq \overline{G}\gamma_{3m}$. We conclude that in the limit $v_1(P) \geqq \min G\gamma_{3m}$, where P is on σ_{3m+2}, and Q on B_{3m}, i. e., $pP = \lambda^{3m+2}$ and $pQ = \leqq \lambda^{3m}$. We find for the minimum of Green's function for a sphere of radius c, when the distances from the center of P and Q do not exceed a and b, respectively, the value

$$\frac{1}{a+b} - \frac{c}{c^2+ab},$$

or, in our present case,

$$\min G \geqq \frac{1}{\lambda^{3m}} \frac{(1-\lambda)(1-\lambda^3)}{(1+\lambda^2)(1+\lambda^4)} = \frac{\alpha}{\lambda^{3m}},$$

where α is independent of m. Hence, for P on or within σ_{3m+2}, $v_1(P) \geqq \alpha\gamma_{3m}/\lambda^{3m}$.

Next, consider the sequence solution $v_2(P)$ corresponding to the points within the same sphere, σ_{3m-1}, except for the *two* sets, B_{3m} and B_{3m+3}, and corresponding to boundary values 0 on σ_{3m-1}, and 1 elsewhere. This function again is dominated by $V_m(P)$, and it dominates $v_1(P)$. Hence $v_2(P) - \alpha\gamma_{3m}/\lambda^{3m} \geqq 0$ on σ_{3m+2} and corresponds to values $1 - \alpha\gamma_{3m}/\lambda^{3m}$ on B_{3m+3}, and so, on σ_{3m+5}, is not less than $1 - \alpha\gamma_{3m}/\lambda^{3m}$ times the function corresponding to $v_1(P)$ with $3m$ replaced by $3m+3$. That is, on and within σ_{3m+5},

$$v_2(P) - \alpha\gamma_{3m}/\lambda^{3m} \geqq (1 - \alpha\gamma_{3m}/\lambda^{3m})\alpha\gamma_{3m+3}/\lambda^{3m+3},$$

or

$$v_2(P) \geqq 1 - (1 - \alpha\gamma_{3m}/\lambda^{3m})(1 - \alpha\gamma_{3m+3}/\lambda^{3m+3}).$$

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* Kellogg, loc. cit., I.

Proceeding in this way, with $v_r(P)$ the sequence solution corresponding to values 0 on σ_{3m-1} and 1 on B_{3m}, B_{3m+3}, $\cdots$, $B_{3(m+r-1)}$, we find that within $\sigma_{3(m+r)-1}$,

$$V_m(P) \geqq V_r(P) \geqq 1 - \prod_{i=0}^{r-1} (1 - \alpha\gamma_{3(m+i)}/\lambda^{3(m+i)}).$$

But since, by hypothesis, the infinite series $\sum_1^\infty (\gamma_{3n}/\lambda^{3n})$ diverges, the infinite product $\prod_1^\infty (1-\alpha\gamma_{3n}/\lambda^{3n})$ diverges to 0, and hence there is a sphere about p within which $V_m(P)$ is arbitrarily close to 1.

This property of $V_m(P)$ being established, it is seen that

$$U_p(P) = \sum_1^\infty \frac{1}{2^m}(1 - V_m(P))$$

is a barrier for T at p, for the series is in T a uniformly convergent series of harmonic functions approaching 0 at p, and elsewhere positive, with positive boundary values.

Suppose now that $\sum_1^\infty (\gamma_n/\lambda^n)$ converges. We shall see that the assumption that p is regular leads to a contradiction. Let m be chosen so that $\sum_m^\infty (\gamma_n/\lambda^n) < \lambda/4$. Then, if p is regular, the conductor potential, $V_m(P)$, of the points within σ_m but not in T, approaches 1 at p, since the sequence solution approaches the assigned values at regular boundary points. Hence there is a sphere, σ', about p, within which $V_m(P) > 3/4$. On the other hand, for sufficiently large k, $V_{m+k}(P)$, being the conductor potential of a set within σ_{m+k}, cannot exceed $1/4$ on σ'. Now the conductor potential of the sum of two point sets never exceeds the sum of the conductor potentials.* Hence, if $V_{m,m+k}$ denotes the conductor potential of the set of points not in T and whose distances from p lie in the closed interval $(\lambda^{m+k+1}, \lambda^m)$, $V_m \leqq V_{m+k} + V_{m,m+k}$, and hence, in particular, on σ', $3/4 < 1/4 + V_{m,m+k}$, that is $V_{m,m+k} > 1/2$.

On the other hand, $V_{m,m+k} \leqq \sum_m^{m+k} v_n$, where v_n is the conductor potential of B_n. But the value of a conductor

* Kellogg, loc. cit., I.

potential is never decreased at an external point, p, by moving the whole charge to the point of the set nearest p, and hence $V_{m,m+k} \leqq \sum_m^{m+k} (\gamma_n/\lambda^{n+1}) < 1/4$. Thus a contradiction has been arrived at, and p must be exceptional.

8. *Removable Singularities.* Another problem which is illuminated by the notion of capacity is that of the removable singularities of bounded harmonic functions. We have seen that in the plane, isolated points, and in space of three dimensions, arcs of curves, can be the seats of no singularities of bounded harmonic functions other than removable ones. The following is the complete generalization of these facts.

Let T denote a domain, and let B denote a sub-set of its boundary with the properties *a*) the set $T' = T + B$ is a domain (open continuum), and *b*) B has capacity 0. Then any function, bounded and harmonic in T, may be so defined on B as to be harmonic in T'. Conversely, if B is a set with the property *a*, and if no bounded harmonic function in T can have singularities at the points of B other than removable ones, then B must have 0 capacity.*

There exists also a certain maximal set B, such that no further boundary points can be added to it without impairing one of the above properties. This is the complete improper set of Bouligand, consisting of all points at which $\lim G(P, Q) > 0$.

We have stated (p. 611) that a regular boundary point of a plane domain must be a point of condensation of the boundary, and the assertion is based upon the fact that a closed denumerable set has capacity 0. This is included in a theorem of Bouligand:† A necessary and sufficient condition that B have capacity 0 is that given any distribution of positive masses on B such that any sphere containing points of B contains a positive total mass, the potential of this distribution shall have the upper limit $+\infty$ at each point of B.

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* Kellogg, loc. cit., I.

† ANNALES DE LA SOCIÉTÉ POLONAISE, loc. cit. p. 103.

For if to each point, P_n, of the denumerable set we assign a mass $1/n^2$, the potential will have the required property.

9. *Uniqueness.* If at any boundary point a harmonic function fails to approach a boundary value continuously, then boundary values no longer determine it uniquely, as Schwarz* first observed. The situation whose existence he pointed out is present in the case of every domain which is not normal. For, as we have seen, there exist functions, positive in such a domain, and approaching 0 at every regular boundary point (p. 615). But these functions may not be bounded. In many cases, the additional requirement that the function be bounded is sufficient to insure uniqueness. The question of the unique determination of harmonic functions by continuous boundary values for non-normal domains has not yet attained a satisfactory solution. The following is a partial result in this direction.

If the exceptional points of a domain T with bounded boundary form a set of capacity 0, then there is one and only one function, bounded and harmonic in T, which assumes preassigned continuous values at every regular boundary point. This is obvious, since the difference of two such functions would be dominated by a constant times the conductor potential of any set enclosing the exceptional points of the boundary of T, i. e., of a set of arbitrarily small capacity. Such a conductor potential, however, can be made arbitrarily small at any given point of T by sufficiently shrinking up the set of which it is the conductor potential. For domains of this character, then, the sequence solution comes as near to solving the Dirichlet problem as any bounded function can.

10. *Discontinuous Boundary Values.* We are primarily concerned in this review with the Dirichlet problem for continuous boundary values. But generalizations in the direction of discontinuous boundary values have been attained which are too significant not to receive attention.

* GESAMMELTE WERKE, vol. 2, p. 194.

The investigations in question started with simple domains. Studying the circle, Fatou* showed that if $f(\theta)$ is periodic, bounded and summable, Poisson's integral formed with $f(\theta)$ approaches $f(\theta)$ along radii at all points of the circumference with possible exception of a set of measure 0. Moreover, no other bounded harmonic function approaches the same boundary values in the same way. These results were extended by the author† to domains of finite connectivity the coordinates of whose boundary as functions of the arc length have second derivatives satisfying uniformly a generalized Lipschitz condition; it was shown also that for approach along a radius may be substituted approach along any curve meeting the boundary orthogonally with finite curvature.

Evans‡ carried the generalization to any domain whose boundary consists of a finite number of connected sets. If $h(P, Q)$ is a conjugate of Green's function, there is, for fixed Q, a one to one correspondence between the accessible boundary points and the values of h between 0 and 2π, with exceptions which do not affect the Lebesgue integral. If the boundary values constitute a bounded summable function of h (and if they do so for one position of the pole, Q, they will for all positions), then there is one and only one function, bounded and harmonic in T with first partial derivatives which are summable in T, which takes on the given values on the boundary for all values of h except for a set of measure 0. This function is given by Green's integral.

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* *Séries trigonométriques et séries de Taylor*, ACTA MATHEMATICA, vol. 30 (1906), p. 339 et seq.

† *Harmonic functions and Green's integral*, TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), pp. 109–132.

‡ *Fundamental points of potential theory*, RICE INSTITUTE PAMPHLET, vol. 7 (1920), pp. 252–329, especially p. 327; *Problems of potential theory*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 7 (1921), pp. 89–98; *The Dirichlet problem for the general finitely connected region*, PROCEEDINGS OF THE TORONTO CONGRESS.

Evans and Bray* have studied Poisson's integral and Green's integral for plane domains of finite connectivity when expressed as Stieltjes integrals with respect to the boundary values, obtaining necessary and sufficient conditions on a harmonic function that it be so expressible. They have announced similar results for the sphere.†

The Dirichlet problem for a general domain and for any bounded boundary values has been treated by Perron‡ and Remak§ in essentially the following way. Let $\overline{F}(p)$ and $\underline{F}(P)$ be the upper and lower limit functions of the assigned boundary values, $F(p)$. A function is called an *upper function* for these boundary values if it is continuous in $T+t$ and superharmonic, and is not less than $\overline{F}(p)$ on the boundary, t. It is a *lower function* if continuous in $T+t$ and subharmonic and not greater than $\underline{F}(p)$ on t. The lower limit, u, of the set of all upper functions for $F(p)$, and the upper limit, v, of the set of all lower functions for $F(p)$ are harmonic in T. By means of barriers, it is shown that if T satisfies a condition employed by Lebesgue¶ (the conclusion holds if T is any normal domain), $\underline{u}(p) \geqq \underline{F}(p)$ and $\bar{u}(p) \leqq \overline{F}(p)$. Hence, in particular, if T is normal and $F(p)$ continuous, u is the solution of the Dirichlet problem.

Furthermore, whether T is normal or not, $F(p)$ being continuous, u and v coincide, and are identical with the

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* *Sur l'intégral de Poisson généralisée*, COMPTES RENDUS, vol. 176 (1923), p. 1042; *La formule de Poisson et le problème de Dirichlet*, COMPTES RENDUS, vol. 176 (1923), p. 1368; also Evans, *Sur l'intégral de Poisson*, COMPTES RENDUS, vol. 177 (1923), p. 241; Evans, *The Dirichlet problem for the general finitely connected region*, PROCEEDINGS OF THE TORONTO CONGRESS.

† This BULLETIN, vol. 29 (1923), p. 210.

‡ *Eine neue Behandlung der ersten Randwertaufgabe für* $\Delta u = 0$, MATHEMATISCHE ZEITSCHRIFT, vol. 18 (1923), pp. 42–54. See also Radó and F. Riesz, *Über die erste Randwertaufgabe für* $\Delta u = 0$, MATHEMATISCHE ZEITSCHRIFT, vol. 22 (1925), pp. 41–44.

§ *Über potentialkonvexe Funktionen*, MATHEMATISCHE ZEITSCHRIFT, vol. 20 (1924), pp. 126–130.

¶ Loc. cit., RENDICONTI DI PALERMO.

sequence solution.* Thus the sequence solution appears stripped of the unessential elements involved in the nested set of domains and the continuous extension to T of the boundary values.

In very simple cases, however, u and v need not coincide; for instance, in the case of the circle for boundary values which are 1 at points corresponding to rational values of the angular coordinate, and 0 elsewhere. Wiener† has considerably extended the class of boundary functions to which a harmonic function can be assigned, in a paper in which a harmonic function is regarded as a homogeneous linear functional of its boundary values. The analytic expression employed for the harmonic function is a form of generalized integral due to Daniell.‡ Broad conditions for the existence of this integral are set up in terms of capacity. If it exists, the harmonic function it represents has upper and lower limits at a boundary point, p, which lie in the closed interval determined by the corresponding limits of the boundary function.

11. *Conclusion.* From the above account, it should appear that the Dirichlet problem has recently been through a period of remarkable development. But it is characteristic of scientific progress that each advance raises new questions. It may not be out of place to indicate here some connected with the Dirichlet problem.

With respect to general domains, there is the question of the frequency of possible exceptional points. Can they form a set of positive capacity? Must every sub-set of a boundary contain regular points, if the sub-set is closed and has positive capacity? Are there domains admitting

* Wiener, *Note on a paper of O. Perron*, JOURNAL OF MATHEMATICS AND PHYSICS, Mass. Inst. of Tech., vol. 4 (1925), pp. 21–32.

† *Discontinuous boundary conditions and the Dirichlet problem*, TRANSACTIONS OF THIS SOCIETY, vol. 23 (1925), pp. 307–314.

‡ *A general form of integral*, ANNALS OF MATHEMATICS, (2), vol. 19 (1917–18), pp. 279–294.

a bounded harmonic function other than 0 which approaches 0 at every regular boundary point?

Further questions arise in the study of boundary values which may be different for the different possible sets of curves of approach to a multiple boundary point, yet which are continuous in the sense that the difference in the boundary values at p and q approaches 0 with the lower limit of the length of the curves lying in T except for their end points, and connecting p and q (these points being accessible). It should not be difficult to study the existence and properties of the sequence solution in this case. To be sure, Evans' results hold for such boundary values. But the domains he has treated so far are normal ones save for possible isolated boundary points.

There are also interesting problems connected with capacity. Particularly needed are further inequalities connecting the capacity of a set with the capacities of its subsets. Is it possible to express conditions for regularity in some even more purely geometric form than those in terms of capacity?*

For the most part, recent investigations have been concerned with non-overlapping domains.† Generalizations in this respect should be considered. Finally, a number of questions suggest themselves in connection with the theory of functions of a complex variable, and the extension of the results gained for Laplace's equation to other differential equations of elliptic type has scarcely begun.‡

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* In this connection, see the interesting note of Bouligand, *Dimension, étendu, et densité*, COMPTES RENDUS, vol. 180 (1925), pp. 246–48.

† But Perron (loc. cit.) admits overlapping, provided the domain contains no branch points. See also Hŭrwitz-Courant, *Funktionentheorie*, 2d ed., Berlin, 1925.

‡ The reader desiring references to the literature besides those given above, will find an excellent bibliography in Bouligand's *Fonctions harmoniques. Principes de Picard et de Dirichlet*, MÉMORIAL DES SCIENCES MATHÉMATIQUES, Paris, Gauthier-Villars, 1926.

ON IRREDUNDANT SETS OF POSTULATES

BY ALONZO CHURCH

In a recent paper* Dr. H. M. Gehman has made the point that each of the irredundant sets of postulates proposed by the author in a previous paper† can be obtained from another set, which is not only not irredundant but not even independent, by the mechanical method given by the author and restated by Gehman in his paper.

This mechanical process converts any independent set of postulates into an irredundant set, and, as pointed out by Gehman, it has the same effect on certain non-independent sets. But, applied to an average set of postulates, this process yields postulates which are more or less complicated mixtures of irrelevant ideas. If an irredundant set of postulates is to be of any interest, the postulates of the set ought not to be such mixtures of irrelevant ideas; but the fact that the set can be obtained by the mechanical process from some other set is not an objection to it.

Indeed, any irredundant set can be considered as obtained by the mechanical method. For let A, B, C be irredundant. In view of the irredundance, B is equivalent to *if* A *then* AB, where AB means A *and* B. This equivalence is strict. It does not depend on the presence of other postulates. Similarly C is equivalent to *if* A *and* AB *then* ABC. Accordingly the set A, B, C can be restated in the form A, *if* A *then* AB, *if* A *and* AB *then* ABC. It is then clear that this set can be obtained by the mechanical method from the set A, AB, ABC. And the latter set is not even independent.

In this way, given any irredundant set, we can give it the form of a set obtained by the mechanical method if we

*This BULLETIN, vol. 32 (1926), pp. 159–161.

†TRANSACTIONS OF THIS SOCIETY, vol. 27 (1925), pp. 318–328.

introduce suitable redundancies into the statements of the postulates.

Gehman has used exactly this device. Using Gehman's abbreviated terminology, the author's postulates for a finite cycle were:

1. An S-set exists.
2. No proper part of N is an S-set.

Gehman restates the second postulate as follows:

B_2. If an S-set exists, N is the only S-set.

This is, of course, simply an abbreviation for:

B_2. If an S-set exists, N is an S-set, and no set other than N is an S-set.

This is highly redundant. The clause "N is an S-set" is superfluous,* and when it is dropped "If an S-set exists" becomes equally so.† As a matter of fact, B_2 is simply the postulate *if* 1 *then* 1 *and* 2, plus additional redundancies.

Gehman proposes to replace 1 and 2 by the single postulate:

A_2. N is the only S-set.

But A_2 clearly ought to be separated into the two postulates:

N is an S-set.

No set other than N is an S-set.

If we weaken the first of these to read "An S-set exists," we have postulates 1 and 2 again.

The postulates proposed by the author for the system of positive and negative integers were the following:

1. An S-set exists.
2. If N is an S-set, it is not the only S-set.
3. If an S-set exists, an ST-set exists.
4. No proper part of N is an ST-set.

Gehman replaces 4 by:

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*Because if an S-set exists, and no set other than N is an S-set, then necessarily N is an S-set.

†Because if an S-set does not exist it follows at once that no set other than N is an S-set.

C_4. If an ST-set exists, N is the only ST-set.

This substitution introduces redundancies in exactly the same way as before. Postulate 3 of this set, however, stands without alteration in the form, *if* 1 *then so and so.* This is possibly a reason for combining 1 and 3 into the single postulate, "An ST-set exists," a procedure which would not, of course, alter the irredundance of the set of postulates. But the retention of 1 and 3 as separate postulates is defensible on the ground that no irrelevancies among the parts of any postulate are thereby introduced.

Gehman has shown that postulates 1, 3, and 4 can be derived from another set by the mechanical process. There is no reason for the omission of postulate 2, because it is equally true that postulates 1, 2, 3, and 4 can be derived from another set by the mechanical process.

For example, arranged in the order 4, 3, 1, 2, they can be derived by the mechanical method from the following postulates:

4°. No proper part of N is an ST-set.

3°. If an S-set exists, N is the only ST-set.

1°. N is the only ST-set.

2°. N is the only ST-set but not the only S-set.

A similar statement is true of the postulates arranged in any other order. In fact, not only can the set of postulates be derived by the mechanical process from a non-independent set, but it can be so derived from each of 4! different sets no one of which is independent. And the same thing is true of any irredundant set of four postulates.

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GENERALIZATION OF LAGRANGE'S THEOREM

BY LOUIS WEISNER

1. *Introduction.* The following theorem due to Lagrange is of considerable importance in the theory of equations.

LAGRANGE'S THEOREM. *If the group to which the rational function* $\psi(x_1, \cdots, x_n)$ *belongs is a subgroup of the group to which the rational function* $\phi(x_1, \cdots, x_n)$ *belongs, then* ϕ *equals a rational function of* ψ *and the elementary symmetric functions of the variables* $x_1, \cdots, x_n$.

In this paper I prove a similar theorem for sets of variables.

2. *Notation and Definitions.* Consider the n sets of m variables $x_{1i}, x_{2i}, \cdots, x_{mi}$ $(i=1, \cdots, n)$, which may be regarded as coordinates of n points in m-space. By a permutation of these sets of variables we mean a permutation of the points. Thus a permutation which changes x_{1i} to x_{1j}, also changes $x_{2i}, \cdots, x_{mi}$ to $x_{2j}, \cdots, x_{mj}$ respectively. It is simpler to regard the permutation as affecting the second subscripts of the variables, with the above notation, than as affecting the x's.

A function $\phi(x_{11}, x_{21}, \cdots, x_{m1}; \cdots; x_{1n}, x_{2n}, \cdots, x_{mn})$ is said to *belong* to a substitution group G on the symbols $1, 2, \cdots, n$, if ϕ is unaltered by every substitution of G and by no substitution on these symbols not contained in G. There exist functions which belong to a given substitution group. In fact, we can construct such functions involving only the variables $x_{11}, x_{12}, \cdots, x_{1n}$.*

3. *A Generalization.* We proceed to prove the following generalization of Lagrange's Theorem.

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* Netto, *Substitutionentheorie und ihre Anwendung auf die Algebra*, 1882, p. 27.

THEOREM. *If the group to which the rational function* $\psi(x_{11}, x_{21}, \cdots, x_{m1}; \cdots; x_{1n}, x_{2n}, \cdots, x_{mn})$ *belongs, is a subgroup of the group to which the rational function* $\phi(x_{11}, x_{21}, \cdots, x_{m1}; \cdots; x_{1n}, x_{2n}, \cdots, x_{mn})$ *belongs, then* ϕ *equals a rational function of* ψ *and the elementary symmetric functions of the sets of variables* $x_{1i}, x_{2i}, \cdots, x_{mi}$, $(i=1, \cdots, n)$.

It will suffice to consider the case $m=3$. The elementary symmetric functions of the n triads of variables are defined by*

$$p_{ijk}=\sum x_{11}x_{12}\cdots x_{1i}\,x_{2,1+i}x_{2,2+i}\cdots x_{2,j+i}x_{3,1+i+j}x_{3,2+i+j}\cdots x_{3,k+i+j} \qquad (i+j+k\leqq n).$$

With the aid of these functions, we can express any one of the variables x_{1i}, x_{2i}, x_{3i} as a rational function of any one of the others. In fact,† we have

$$x_{1i}=\frac{p_{100}x_{3i}^{n-1}-p_{101}x_{3i}^{n-2}+p_{102}x_{3i}^{n-3}-\cdots}{nx_{3i}^{n-1}-(n-1)p_{001}x_{3i}^{n-2}+(n-2)p_{002}x_{3i}^{n-2}-\cdots},$$

$$x_{2i}=\frac{p_{010}x_{3i}^{n-1}-p_{011}x_{3i}^{n-2}+p_{012}x_{3i}^{n-3}-\cdots}{nx_{3i}^{n-1}-(n-1)p_{001}x_{3i}^{n-2}+(n-2)p_{002}x_{3i}^{n-2}-\cdots}.$$

Hence every function of the triads of variables can be expressed as a function of $x_{21}, x_{32}, \cdots, x_{3n}$, with coefficients that belong to the symmetric group. In particular, suppose

$$\psi(x_{11},x_{21},x_{31};\cdots;x_{1n},x_{2n},x_{3n})=\psi_1(x_{31},x_{32},\cdots,x_{3n}),$$

$$\phi(x_{11},x_{21},x_{31};\cdots;x_{1n},x_{2n},x_{3n})=\phi_1(x_{31},x_{32},\cdots,x_{3n}).$$

Evidently ψ and ψ_1 belong to the same group H, and ϕ and ϕ_1 belong to the same group G. As H is a subgroup of G by hypothesis, it follows from Lagrange's Theorem, that ϕ equals a rational function of ψ and the elementary symmetric functions $p_{001}, p_{002}, \cdots, p_{00n}$. The theorem follows.

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* See Bôcher, *Higher Algebra*, p. 252.

† Netto, *Vorlesungen über Algebra*, vol. II, p. 71.

THE TRANSFORMATION OF A REGULAR GROUP INTO ITS CONJOINT

BY G. A. MILLER

The main purposes of the present article are to correct an assumption which appeared in a former article with the same title,* and to exhibit in a definite manner a method of finding a particular substitution of order 2 which transforms a given regular substitution group into its conjoint. It is well known that a regular group R_1 which is simply isomorphic with an arbitrary finite group G whose operators are $1, s_2, s_3, \cdots, s_g$ can be obtained by letting $s_\alpha(\alpha=2, 3, \cdots, g)$ correspond to the permutation of the operators of G when all of these operators are multiplied on the right by s_α. Another simply isomorphic regular group R_2 may be obtained by letting s_α correspond to the permutation of these operators when all of them are multiplied on the left of s_α^{-1}. From the associative law it results that every substitution of R_1 is commutative with every substitution of R_2, and hence R_1 and R_2 are the conjoint of each other.†

From the preceding paragraph it results that any regular group and its conjoint can be made simply isomorphic as follows: Make all the cycles involving a particular letter of the regular group correspond to their inverses in the conjoint group, and adjoin to each such inverse its conjugates under the given regular group. A necessary and sufficient condition that a substitution corresponds to its inverse in this simple isomorphism is that it is invariant under the given regular group. In particular, all the substitutions of this group will thus correspond to their inverses when the

* This BULLETIN, vol. 25 (1919), p. 326.

† W. Burnside, *The Theory of Groups*, 1911, p. 88.

group is abelian, and only then. It is important to observe that a substitution t of order 2 which transforms the given regular group into its conjoint can be obtained by letting the common letter of the said inverse cycles correspond to itself and noting the permutation of the remaining letters involved in these corresponding cycles. The number of letters omitted by t is just one larger than the number of substitutions of order 2 found in the regular group.

In the article to which reference was made in the first paragraph it is assumed, page 328, that in some simple isomorphism between every regular group and its conjoint every two corresponding substitutions involve a common cycle. From the results of the preceding paragraph it is easy to see that it is sometimes impossible to establish such a simple isomorphism between these groups. For instance, when R_1 is the non-cyclic group of order 21 it is obvious that the cycles of order 3 found in a co-set with respect to the subgroup of order 7 include all their conjugates under the group, and that one of these co-sets cannot be transformed into the other by an operator of the holomorph of the group. Hence it results that no two corresponding substitutions of order 3 in a simple isomorphism between this group and its conjoint can involve a common cycle. On the other hand it has been proved that *a simple isomorphism between a regular group and its conjoint can always be established in such a manner that at least two cycles in every pair of corresponding substitutions are inverses of each other.*

The substitution t noted above which transforms R_1 and R_2 into each other is commutative with every substitution of the subgroup composed of all the substitutions of the double holomorph of R_1 (when R_1 is non-abelian) which omit the letter common to the cycles of R_1 which were made to correspond to their inverses in the given simple isomorphism between R_1 and R_2. This results from the facts that t transforms each of these cycles into its inverse and the totality of these cycles is transformed into itself by

all the substitutions of the subgroup in question. In particular, we have established the theorem that *every regular group can be transformed into its conjoint by an operator of order 2 which is commutative with every operator of the group of isomorphisms of this regular group.* When R_1 is non-abelian its double holomorph is composed of all the substitutions on the letters of R_1 which transform R_1 either into itself or into R_2. The subgroup composed of all the substitutions of this double holomorph which omit a given letter thereof is therefore the direct product of the group of isomorphisms of R_1 and a substitution of order 2.

The number of letters in the cycles which correspond to their inverses in a particular pair of corresponding substitutions in the simple isomorphism between R_1 and R_2 obtained by transforming R_1 by t is equal to the number of the substitutions of R_1 which are commutative with the substitution of R_1 involving these cycles. In particular, all of these cycles appear in the same substitutions of R_1. In the special case when R_1 is a dihedral group whose order is twice an odd number only one cycle of order 2 corresponds to its inverse in such an isomorphism, while two such cycles correspond to their inverses in the non-invariant substitutions of order 2 when the order of the dihedral group is divisible by 4 and exceeds 4. In each of these cases t is of a smaller degree than any substitution besides the identity found in the holomorph of the dihedral group in question. A necessary and sufficient condition that t omits only one letter of R_1 is that the order of R_1 is odd.

When the group G contains no invariant operator besides the identity the substitution t may be exhibited as follows: Represent G as a substitution group and write it similarly on two distinct sets of letters. Extend the direct product of the conjugate groups thus obtained by a substitution s of order 2 which merely interchanges their corresponding letters. The subgroup of index g formed by extending by

means of s the simple isomorphism between these two groups, in which all substitutions are commutative with s, involves no invariant subgroup of the entire group besides the identity and is not invariant. Hence the entire group can be represented as a transitive group of degree g with respect to this subgroup. When it is thus represented the factors of the said direct product will be a regular group and its conjoint respectively, and s will obviously be the substitution t as regards these groups. This process exhibits also clearly some of the properties of a group and its conjoint when the group involves no invariant operator besides the identity.

When G is a complete group this process clearly gives the double holomorph of G. Hence it throws light on the difficult problem of constructing a complete group of odd order. The subgroup composed of all the substitutions which omit one letter of the holomorph of such a group must be of degree $g-1$ and its transitive constituents can be combined into pairs which are transformed into each other by t. Moreover, *every substitution which transforms a regular complete group of odd order g into its conjoint is of degree $g-1$.* In particular, such a group cannot be made simply isomorphic with its conjoint in such a way that some two corresponding substitutions involve a common cycle.

THE UNIVERSITY OF ILLINOIS

A CHARACTERISTIC PROPERTY OF MINIMAL SURFACES*

BY JESSE DOUGLAS†

1. *Statement of Theorems.* By the mean curvature of a surface at a point we shall understand $1/R_1+1/R_2$, where $1/R_1$, $1/R_2$ are the principal curvatures at the point. There is some divergence in the literature with respect to this nomenclature; some authors call $(1/R_1+1/R_2)/2$ the mean curvature.‡

Imagine that about any point O of an arbitrary surface Σ as center we describe a sphere S of infinitesimal radius r. Then S is intersected by Σ in a curve C infinitely close§ to the great circle cut from S by the tangent plane to Σ at O. The curve C divides the surface of S into two areas nearly equal to each other. Also, the portion of Σ lying within S divides S into two nearly equal volumes.

The present paper is devoted to the proofs of the following three theorems.

THEOREM I. *Let the difference between the two areas into which C divides S be denoted by η. Then, in the limit as $r\to 0$,*

$$\frac{\eta}{V}=\frac{3}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

where V denotes the volume of S.

THEOREM II. *Let v denote the difference of the two volumes into which S is divided by Σ. Then, in the limit as $r\to 0$,*

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* Presented to the Society, October 30, 1926.

† National Research Fellow in Mathematics.

‡ Bianchi, Eisenhart, and Scheffers use the first notation; Blaschke and Darboux the second.

§ Even in comparison with the radius of the sphere, that is, the deviation of C from the great circle is an infinitesimal of the second or higher order.

$$\frac{v}{rV} = \frac{3}{16}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

When Σ is a minimal surface, or one of mean curvature zero, these theorems yield the following corollary.

THEOREM III. *A minimal surface bisects the area and volume of an infinitesimal sphere about any point of the surface as center.*

The sense of this statement is that the bisection occurs to infinitesimals of higher order for a minimal surface than for a non-minimal surface.

2. *Proof of Theorem* I. The proof of these theorems is a straightforward problem in calculus. We choose for axes the tangents to the lines of curvature at O and the normal to the surface. Then, for all our purposes, the surface may be replaced by the paraboloid

$$z = ax^2 + by^2 \tag{1}$$

having second order contact with it at O. Here

$$a = \frac{1}{2R_1}, \quad b = \frac{1}{2R_2}. \tag{2}$$

The equation of the sphere S is

$$x^2 + y^2 + z^2 = r^2. \tag{3}$$

We use cylindrical coordinates ρ, θ, z, in terms of which the equations of Σ, S are respectively

$$z = \rho^2(a\cos^2\theta + b\sin^2\theta), \tag{4}$$

$$\rho^2 + z^2 = r^2. \tag{5}$$

The polar equation of the projection C_1 of C on the tangent plane at O is

$$\rho_1^2 + \rho_1^4(a\cos^2\theta + b\sin^2\theta)^2 = r^2.$$

Expanding ρ_1 in powers of r, we need retain only the first two terms, and have

$$\rho_1 = r - \tfrac{1}{2}r^3(a\cos^2\theta + b\sin^2\theta)^2 \tag{6}$$

for the (approximate) polar equation of C_1. The area of the portion of the sphere above C is

$$A_1 = 4\int_0^{\pi/2}\int_0^{\rho_1} \frac{r}{(r^2-\rho^2)^{1/2}}\rho d\rho d\theta.$$

Performing the first integration, we get

$$A_1 = 4r\int_0^{\pi/2} \left\{-(r^2-\rho^2)^{1/2}\right\}_0^{\rho_1} d\theta.$$

Now we have

$$\left\{-(r^2-\rho^2)^{1/2}\right\}_0^{\rho_1} = r - r^2(a\cos^2\theta + b\sin^2\theta)$$

where, as will be our rule hereafter, we neglect all infinitesimals beyond those which would influence the final result. Performing the second integration we find

$$A_1 = 2\pi r^2 - \pi r^3(a+b).$$

The area of the portion of the sphere to the other side of C is the complement of A_1 as to $4\pi r^2$, or

$$A_2 = 2\pi r^2 + \pi r^3(a+b).$$

The difference

$$\eta = A_2 - A_1 = 2\pi r^3(a+b) = \pi r^3\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

by (2). Introducing $V = \frac{4}{3}\pi r^3$, we have

$$\frac{\eta}{V} = \frac{3}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

or Theorem I.

3. *Proof of Theorem* II. We first obtain the volume V_1 above Σ and below the sphere by calculating the difference $V_1 = V_S - V_\Sigma$, where

V_S = volume under portion of sphere above C,

V_Σ = volume under portion of Σ within C.

The volume V_S turns out to be that of a hemisphere up to infinitesimals of the sixth order. For, with the use of (5) and (6),

$$V_S = 4\int_0^{\pi/2}\int_0^{\rho_1}(r^2-\rho^2)^{1/2}\rho d\rho d\theta$$

$$= 4\int_0^{\pi/2}\tfrac{1}{3}\left\{r^3-(r^2-\rho_1^2)^{3/2}\right\}d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\left\{r^3-r^6(a\cos^2\theta+b\sin^2\theta)^3\right\}d\theta = \frac{2}{3}\pi r^3,$$

neglecting the term in r^6.

$$V_\Sigma = \iint_{\Sigma_1} z\rho d\rho d\theta$$

where Σ_1 is the area enclosed by C_1, the projection of C. By (4),

$$V_\Sigma = 4\int_0^{\pi/2}\int_0^{\rho_1}\rho^3(a\cos^2\theta+b\sin^2\theta)d\rho d\theta$$

$$= \int_0^{\pi/2}\rho_1^4(a\cos^2\theta+b\sin^2\theta)d\theta.$$

By (6), $\rho_1^4 = r^4$ up to terms in r^6 and higher powers. Therefore $V_\Sigma = \pi r^4(a+b)/4$. It follows that

$$V_1 = \frac{2}{3}\pi r^3 - \frac{\pi}{4}r^4(a+b).$$

The complementary volume of V_1 as to the sphere is

$$V_2 = \frac{2}{3}\pi r^3 + \frac{\pi}{4}r^4(a+b).$$

Dividing the difference $v = V_2 - V_1$ by $rV = 4\pi r^4/3$, we obtain

$$\frac{v}{rV} = \frac{3}{8}(a+b) = \frac{3}{16}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

that is, Theorem II.

PRINCETON UNIVERSITY

ON THE CONVERGENCE OF TRIGONOMETRIC APPROXIMATIONS FOR A FUNCTION OF TWO VARIABLES*

BY ELIZABETH CARLSON

A discussion of the convergence of approximating functions for a given function of two variables $f(x, y)$, just as in the case of functions of one variable, can be based on two sets of theorems: (1) theorems on the existence of functions of closest approximation; (2) theorems on the representation of $f(x, y)$ by means of finite sums constructed in a specific way.

From the first group we shall make use of the following theorem:

THEOREM I. *Let $p_1(x,y)$, $p_2(x\,y)$, $\cdots$, $p_N(x,y)$ be N functions of x and y, continuous in the region R: $(a \leqq x \leqq b$, $c \leqq y \leqq d)$, and linearly independent in this region. Let*

$$\phi(x,y) = c_1p_1(x,y) + c_2p_2(x,y) + \cdots + c_Np_N(x,y)$$

be an arbitrary linear combination of the given functions with constant coefficients. Let $f(x, y)$ be a continuous function of x and y in R. Then there exists a choice of the coefficients c_k in $\phi(x, y)$ such that the integral

$$\int_a^b dx \int_c^d |f(x,y) - \phi(x,y)|^m dy, \qquad (m > 0),$$

has its minimum value. The function $\phi(x, y)$ so determined is unique for $m > 1$. It is called an approximating function for $f(x, y)$ corresponding to the exponent m.

This theorem can be proved by methods analogous to those used in proving the corresponding theorem for functions of a single variable.† In this paper we shall choose

* Presented to the Society, April 14, 1922.

† Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128; *Note on an ambiguous case of approximation*, TRANSACTIONS, vol. 25 (1923), pp. 333–337.

the p's so that a function of the form $\phi(x, y)$ with arbitrary coefficients is an arbitrary trigonometric sum of order n in x and of the same order in y, and we shall take as the region R the square $0 \leqq x \leqq 2\pi$, $0 \leqq y \leqq 2\pi$.

From the second group of theorems, we shall apply the following theorems.

THEOREM II. *If $f(x, y)$, of period 2π in each argument, has a modulus of continuity $\omega(\delta)$, then $f(x, y)$ can be represented everywhere by a trigonometric sum $t_n(x, y)$, of order n in each variable, with an error that does not exceed a constant times $\omega(2\pi/n)$.*

THEOREM III. *If $f(x, y)$ has continuous first partial derivatives, it can be approximately represented with a maximum error ϵ_n such that $\lim_{n=\infty} n\epsilon_n = 0$.*

The assertion II can be proved once more by an adaptation of the reasoning used in the case of functions of a single variable.* A proof of III was communicated to the writer orally by Professor Jackson.

Theorems I and II lead to the following theorem for the convergence of $T_n(x, y)$, the trigonometric approximating function, of order n in each variable, corresponding to the exponent m $(m > 1)$.

THEOREM IV. *If $f(x, y)$, of period 2π in each argument, is everywhere continuous, with a modulus of continuity $\omega(\delta)$, then a sufficient condition for the uniform convergence of $T_n(x, y)$ to $f(x, y)$ is*†

$$\lim_{\delta=0} \omega(\delta)/\delta^{2/m} = 0.$$

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* Jackson, *On approximation by trigonometric sums and polynomials*, TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), pp. 491–515. Cf. C. E. Wilder, *On the degree of approximation to discontinuous functions by trigonometric sums*, RENDICONTI DI PALERMO, vol. 39 (1915), pp. 345–361.

† Cf. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 158–166.

The occurrence of the exponent $2/m$, where $1/m$ is found in the case of functions of a single variable, is due to the fact that the magnitude of an interval of length $1/n$ is replaced in the course of the present reasoning by the *area* of a square having a quantity of the order of $1/n$ for the length of its side.

This result is significant only if $m>2$, since otherwise the condition as stated requires that $f(x, y)$ be constant. *When* $m=2$, *it is sufficient* (in consequence of III) *that* $f(x, y)$ *have continuous first partial derivatives.* There are corresponding results for $m<2$.

The reasoning is not materially changed if the mth power in the integral to be minimized is multiplied by a positive measurable weight function $\rho(x, y)$ with a positive lower bound.

THE UNIVERSITY OF MINNESOTA

ON A TENSOR OF THE SECOND RANK IN FUNCTION SPACE*

BY DUNHAM JACKSON

In a recent paper† the writer has discussed a doubly infinite matrix of derivatives which has the properties of a tensor of the second rank in function space, and a quantity obtained by contraction of this tensor, which is analogous to a divergence. The latter concept is suggested formally by the writing of an infinite series, the general term of which in many cases does not approach zero; but an example was constructed in which the series is convergent, and defines a quantity which can be alternatively expressed in a form independent of any particular coordinate system.

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* Presented to the Society, September 9, 1926.

† D. Jackson, *Some convergence proofs in the vector analysis of function space*, ANNALS OF MATHEMATICS, vol. 27, pp. 551–567.

The purpose of this note is to point out that the tensor of the second rank may appropriately be identified with a certain double series, or with the function of two variables represented by the series, and that this immediately leads to a simplified invariant interpretation of the divergence.

The notation and hypotheses of the earlier paper will be retained. The relation

$$\Phi(x) = \int_0^{2\pi} K(x, t)\phi[f(t)]dt$$

defines what may be regarded as an example of a vector point function in function space, with $f(x)$ as independent variable. The functions

$$u_n(x), \qquad (n=1, 2, \cdots),$$

form a normalized orthogonal system, constructed in a particular way, and having properties which were discussed in some detail in the paper cited. If a_n and A_n are the coefficients of $u_n(x)$ in the developments of $f(x)$ and $\Phi(x)$ respectively according to the system of u's, each A_i is a function of $a_1, a_2, \cdots$, and the general component of the tensor in question is

$$\frac{\partial A_i}{\partial a_j} = \int_0^{2\pi}\int_0^{2\pi} K(x, t)\phi'[f(t)]u_i(x)u_j(t)dt\,dx.$$

The point to be made in the present note is that the last expression is the coefficient of $u_i(x)\, u_j(t)$ in the formal expansion of*

$$K(x, t)\phi'[f(t)] \tag{1}$$

in a double series of the u's. Since it is known furthermore under the hypotheses of § 8 of the paper cited that

$$\left|\frac{\partial A_i}{\partial a_j}\right| \leqq \frac{Q}{i^2j^2},$$

* I am indebted to Dr. A. D. Michal for the observation, which had escaped me when the note was presented to the Society, that for each value of x the expression (1) is the functional derivative of the functional Φ.

where Q is independent of i and j, and since $|u_n(x)|$ does not exceed a constant multiple of $n^{1/2}$, the typical form of the series satisfies an inequality of the form

$$\left|\frac{\partial A_i}{\partial a_j} u_i(x) u_j(t)\right| \leqq \frac{Q'}{i^{3/2} j^{3/2}},$$

and the series converges uniformly, inasmuch as the double series having the quantity on the right for its general term is convergent. A familiar line of argument shows that the sum of the series is necessarily the function (1). This function of two variables may therefore be taken as an invariant representation of the tensor, while the expansion in series indicates its resolution into components according to the orthogonal system.

Termwise integration of the series for $K(t, t)\phi'[f(t)]$, with due regard to the orthogonality of the u's, leads to the relation

$$\sum_i \frac{\partial A_i}{\partial a_i} = \int_0^{2\pi} K(t, t)\phi'[f(t)]dt,$$

which expresses the "divergence" in a form independent of the set of u's taken as a system of reference. The form given in the earlier paper, in terms of double integrals, can be reduced to this one by integration by parts.

THE UNIVERSITY OF MINNESOTA

ALL INTEGRAL SOLUTIONS OF $ax^2+bxy+cy^2=w_1 w_2 \cdots w_n$*

BY L. E. DICKSON

1. *Literature.* This Diophantine equation (or cases of it) has been treated in two papers by the writer and two by Professor Wahlin, all published in this BULLETIN†. Three of these papers were based on the theory of algebraic ideals. The writer's paper of 1923 employed an elementary method to find all integral solutions of $x^2-my^2=zw$. The present paper is elementary and is a sequel to the latter paper.

2. *Reduction to the Case $n=2$.* Let q denote a quadratic form in x and y. The problem to solve $q=zw$ shall be called the homogeneous problem. To it will be reduced the problem to solve $q=w_1 \cdots w_n$. Write $z=w_1 \cdots w_{n-1}$. By our solution below of the homogeneous problem $q=zw_n$, x, y, z, w_n are products of an arbitrary integer h by certain functions X, Y, Z, W of certain parameters, only two of which, say ξ and η, occur in the quadratic expression $Q(\xi, \eta)$ for Z. Since $w_1 \cdots w_{n-1}=hZ$, evidently $w_i=h_iW_i$ $(i=1, \cdots, n-1)$, where the h_i are integers whose product is h. Hence $q=w_1 \cdots w_n$ is reduced to the solution of $Q(\xi, \eta)=W_1 \cdots W_{n-1}$, which is of the form of our initial equation with n replaced by $n-1$. The resulting values of ξ and η in terms of new parameters are to be inserted in the functions X, Y, and W.

3. *Simplification of the Homogeneous Problem.* The greatest common divisor ω of the coefficients of $q(x, y)$ must

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* Presented to the Society, September 8, 1926.

† Vol. 27 (1920–1), p. 361; vol. 29 (1923), p. 464; vol. 30 (1924), p. 140; vol. 31 (1925), p. 430.

divide zw. Hence we may write $z=\rho Z$, $w=\sigma W$, where $\rho\sigma=\omega$. Cancellation of ω gives

$$Q \equiv Ax^2 + Bxy + Cy^2 = ZW, \tag{1}$$

where the greatest common divisor of A, B, C is unity. By Dirichlet's theorem, Q represents an infinitude of primes. Let a be one such odd prime and let $A\kappa^2+B\kappa\lambda+C\lambda^2=a$. Evidently κ and λ are relatively prime. Hence there exist integers μ and ν for which $\kappa\nu-\lambda\mu=1$. Then

$$x = \kappa X + \mu Y, \qquad y = \lambda X + \nu Y \tag{2}$$

is of determinant unity and transforms Q into a form in which the coefficient of X^2 is a. Thus (1) becomes

$$aX^2 + 2bXY + cY^2 = ZW. \tag{3}$$

We shall exhibit sets of functions X, Y, Z, W which give all solutions of (3). Insertion in (2) yields like functions giving all solutions of (1).

After removing a common factor, we may assume that the greatest common divisor of X, Y, Z, W is unity. We may write

$$X = dX_1, \quad Y = dY_1, \quad Z = \delta Z', \quad d = \delta D,$$

where X_1 and Y_1 are relatively prime, and likewise Z' and D. Then δ is prime to W, and

$$d^2(aX_1^2 + 2bX_1Y_1 + cY_1^2) = \delta Z'W.$$

Hence δD^2 divides $Z'W$, so that δ divides Z', and D^2 divides W. Write $Z'=\delta Z_1$, $W=D^2W_1$. Hence

$$aX_1^2 + 2bX_1Y_1 + cY_1^2 = Z_1W_1, \tag{4}$$

$$X = \delta DX_1, \quad Y = \delta DY_1, \quad Z = \delta^2 Z_1, \quad W = D^2W_1. \tag{5}$$

4. *Method of Solving* (4). Multiply (4) by a and write

$$m = b^2 - ac, \quad \xi = aX_1 + bY_1, \quad u = aW_1. \tag{6}$$

We get $\xi^2 - mY_1^2 = Z_1 u$. By the writer's paper in this BULLETIN (vol. 29 (1923), p. 464), we have

$$\xi = hx, \quad Y_1 = hy, \quad Z_1 = hz, \quad u = hw, \tag{7}$$

$$z = el^2 + 2flq + gq^2, \quad w = en^2 - 2fnr + gr^2, \tag{8}$$

$$x = k(eln + fnq - flr - gqr), \quad y = lr + nq, \tag{9}$$

where $k^2 = 1$ and h, l, q, n, r are arbitrary integers, while e, f, g take the finite sets of integral values for which the resulting forms z in (8) include one and only one form from each class of equivalent quadratic forms of discriminant $4m$, whence

$$f^2 - eg = m. \tag{10}$$

Since h divides ξ and Y_1, it divides aX_1. But X_1 and Y_1 are relatively prime. Hence h divides the prime a. Thus $h = \pm 1$ or $\pm a$.

5. *Solutions of* (4) *with* $h = \pm a$. By (5)–(7), we have

$$X = \pm \delta D(x - by), \quad Y = \pm \delta Day, \quad Z = \pm \delta^2 az, \quad W = \pm D^2 w.$$

But if we multiply l and q in (8) and (9) by δ, and n and r by D, we see that z is multiplied by δ^2, w by D^2, and both x and y by δD. Hence the suppression of the factors δD, δ^2, D^2 is equivalent to a change of parameters l, q, n, r. Also the factor ± 1 may be combined with the common factor initially removed from X, Y, Z, W. Hence every solution of (3) with $h = \pm a$ is given by

$$X = s(x - by), \quad P = say, \quad Z = saz, \quad W = sw, \tag{11}$$

where s is an arbitrary integer and x, y, z, w are defined by (8) and (9).

6. *Solutions of* (4) *with* $h = \pm 1$. As in § 5, we may take $h = +1$. Then (6) and (7) give

$$aX_1 + bY_1 = x, \quad Y_1 = y, \quad Z_1 = z, \quad aW_1 = w. \tag{12}$$

The first step is to obtain an integral form of the quotient of w by a. We may replace w in (8) by an equivalent form whose first coefficient is not divisible by a. This is evident unless $e \equiv 0$, $g \equiv 0 \pmod{a}$. Then if f is not divisible by a, we replace r by $r+n$. The case in which also $f \equiv 0 \pmod{a}$ may be excluded* on the ground that m then has the square factor a^2.

Let therefore e be not divisible by the odd prime a. Determine integers E and ϵ so that

$$eE = 1 + \epsilon a. \tag{13}$$

Write $t = en - fr$. By (10), $ew \equiv 0 \pmod{a}$ if and only if $t^2 \equiv mr^2$, and hence by (6) if and only if $t \equiv \pm br \pmod{a}$. Then by (13),

$$n = E(f \pm b)r + av, \tag{14}$$

where v is an integer. Elimination of n from (8_2) gives

$$w = a^2vB + aE(f \pm b)Br - a(f \mp b)vr + Sr^2,$$

where

$$B = \epsilon(f \pm b)r + ev, \qquad S = g - E(f^2 - b^2).$$

By the two values of m in (6) and (10), and by (13),

$$E(f^2 - b^2) = E(eg - ac) = g - aT, \; T = Ec - \epsilon g, \; S = aT. \tag{15}$$

Since all terms of w are now divisible by a, we get †

$$W_1 = aev^2 + 2[a\epsilon(f \pm b) \pm b]vr + [T + E\epsilon(f \pm b)^2]r^2. \tag{16}$$

In the first two equations (12), we insert the value (14) of n. Thus we find

$$Y_1 = qav + [l + qE(f \pm b)]r. \tag{17}$$

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* Or we may treat this case very simply by noting that the coefficients of w and x in (8) and (9) are now all divisible by a, while b is divisible by a by (6) and (10), whence (12) determine X_1 and W_1 integrally.

† The discriminant of this quadratic form in v and r is found to be $4m$.

If $k=\pm 1$ in (9), where the sign is the same as in (14), we find that $x-bY_1$ becomes divisible by a when we apply (13) and (15), whence

$$\pm X_1 = (el+fq\mp bq)v+[el(f\pm b)-qT]r. \tag{18}$$

But if $k=\mp 1$, we get

$$\mp X_1 = (el+fq\pm bq)v+le(f\pm b)r+rP/a, \tag{19}$$

$$P = Eq(f\pm b)^2-gq\pm 2lb.$$

If $b=\beta a$, (6) shows that m is a multiple Ma of a. Then (10) and (13) give $EMa=Ef^2-g(1+ea)$, whence

$$P/a = Eq(\beta^2 a\pm 2f\beta+M)\pm 2l\beta+qge. \tag{20}$$

Finally, let b be not divisible by the odd prime a. Then rP is divisible by a either when $r=Ra$ or when the congruence $P\equiv 0 \pmod a$ is satisfied by choice of l as a linear function of q and a new parameter L. This value of l or the value Ra of r is to be inserted in (16), (17), (19), and $Z_1=z$ of (8).

If we multiply v and r (or R) by D, and q and l (or L) by δ, we see that Z_1 is multiplied by δ^2, W_1 by D^2 and both X_1 and Y_1 by δD. Hence the suppression of the factors δD, δ^2, and D^2 from (5) is equivalent to a change of parameters v, r, etc. Hence every solution of (3) with $h^2=1$ is obtained by multiplying an arbitrary integer by (16), (17), z of (8), and (18) or (19) with P or l or r replaced by the expressions just obtained.

THE UNIVERSITY OF CHICAGO

NUCLEAR POINTS IN THE THEORY OF ABSTRACT SETS*

BY W. SIERPINSKI

1. *Introduction.* E. W. Chittenden has published an interesting note† in which he gave necessary and sufficient conditions that a set E contained in a class (V) (of Fréchet) be perfectly compact, or perfectly self-compact.‡

Chittenden proves the following theorems:

THEOREM I. *If an infinite set E is perfectly compact, E determines at least one nuclear point.*

COROLLARY. *Every set E which is perfectly self-compact contains a nuclear point.*

THEOREM II. *If every infinite subset of a set E of points of the space P determines a nuclear point then E is perfectly compact.*

THEOREM III. *A necessary and sufficient condition that a set E be perfectly compact is that every infinite subset of E determine at least one nuclear point.*

COROLLARY. *A necessary and sufficient condition that every compact set E be perfectly compact is that every infinite compact set E possess at least one nuclear point.*

The proofs of Chittenden are based on the following assumption (§ 3): "Let Q be an aggregate of power μ and of elements q. Of the transfinite ordinals Ω of which the aggregate of all ordinals $\alpha<\Omega$ has the power μ there is a least, Ω_0. Let

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* Presented to the Society, September 9, 1926.

† This BULLETIN, vol. 30 (1924), p. 511.

‡ For the definitions of the terms *class (V)*, *monotone sequence*, *compact*, *perfectly compact*, *perfectly self-compact*, *limit point*, *nuclear point*, see E. W. Chittenden, loc. cit. §§2, 4. A topological space in which every infinite set determines a nuclear point is called by P. Alexandroff and P. Urysohn a *bicompact* space (see MATHEMATISCHE ANNALEN, vol. 92 (1924), p. 260).

(1) $q_1, q_2, q_3, \cdots, q_n, \cdots, q_\omega, \cdots, q_\alpha, \cdots, \quad (\alpha < \Omega_0)$

represent a 1-1 correspondence between the aggregate Q and the aggregate of all ordinal numbers $\alpha < \Omega_0$. *If a sequence of ordinals $\beta < \Omega_0$ is determined so that for every α there is a $\beta > \alpha$, then the β's form a series of the ordinal type Ω_0 and the aggregate of all such ordinals β is of power μ.*"

This assumption is equivalent to the assumption that μ is a *regular* aleph. Since not every aleph is regular, the theorems of Chittenden are proved only for the sets E whose power is a regular aleph.

In this note I shall examine the question: Are the theorems of Chittenden true for *every* set E without any restriction?

2. *Theorem* I *may be False.* Theorem I may be false for a set E whose power is an irregular aleph, e. g., for

$$\aleph_\omega = \sum_{n=0}^{\infty} \aleph_n.$$

Let E be the set of all ordinals $\xi < \omega_\omega$ (ω_ω denoting the least ordinal of power $\aleph_\omega$). Let H be the set of all the sequences $\xi_1, \xi_2, \xi_3, \cdots$ made up of different elements of E. Let $P = E + H$; the neighborhoods are defined as follows.

Let ξ be any element of E and let $\lambda < \xi$; the set of all ordinals η such that $\lambda < \eta \leqq \xi$ is called a neighborhood of ξ. Let $\alpha = (\xi_1, \xi_2, \xi_3, \cdots)$ be any element of H; we call neighborhood of α any set composed of α and $\xi_n, \xi_{n+1}, \xi_{n+2}, \cdots$ $(n = 1, 2, 3, \cdots)$.

Evidently every neighborhood of an element belonging to E contains less than $\aleph_\omega$ elements of P, and every neighborhood of an element belonging to H contains $\aleph_0$ elements of P. Hence every neighborhood contains less than $\aleph_\omega$ elements. Thus, there does not exist any nuclear point of the set E (as the power of E is $\aleph_\omega$).

I shall prove that E is perfectly compact. Let $S = \{G\}$ be any sequence of decreasing subsets of E. Let S_0 be a well-ordered sequence "confinal with S" (i. e. S_0 is con-

tained in S and for each set G belonging to S there exists a set of S_0 contained in G).* The sequence S_0 may be supposed of the least possible ordinal number. Moreover, we may suppose that S does not contain a last element. Hence the ordinal number of S_0 is an "initial" ordinal number ω_n, where n is either 0 or a natural number. Evidently we may assume $S=S_0$. Consider two cases.

CASE 1. $n=0$. Hence $S=(G_1, G_2, G_3, \cdots)$. For each n, we may select an element α_n of G_n that does not belong to G_{n+1}. The aggregate $\alpha=(\alpha_1, \alpha_2, \alpha_3, \cdots)$ is an element of H and is a limit point of any sequence $\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \cdots$. As each element of the sequence $\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \cdots$ belongs to G_n, α is a limit point of G_n $(n=1, 2, 3, \cdots)$. Hence α belongs to G' for each G of S.

CASE 2. $n\geqq 1$. In this case the set $S=\{G_\xi\}$ is not enumerable. Select, for each ξ, an element p_ξ of G_ξ that does not belong to $G_{\xi+1}$ (hence p_ξ does not belong to G_η if $\eta>\xi$). The set Q of all p_ξ is of power $\aleph_n$.

Let E_k be the set of the ordinals $\xi\leqq\omega_k$. Hence $E=E_1+E_2+\cdots$. Hence there exists an index k such that Q contains $\aleph_n$ elements of E_k (otherwise Q would contain $\leqq\aleph_{n-1}$ elements of each E_k and Q would be of power $\leqq\aleph_{n-1}\cdot\aleph_0=\aleph_{n-1}<\aleph_n$, contrary to hypothesis). Let these elements be

$$(2)\qquad p_{\xi_1}, p_{\xi_2}, \cdots, p_{\xi_\omega}, \cdots, p_{\xi_\alpha}, \cdots, \qquad (\alpha<\omega_n).$$

Let γ be the least number having the property that there exists in the sequence (2) a set of power $\aleph_n$ of numbers $\leqq\gamma$. As the sequence (2) is contained in E_k, $\gamma\leqq\omega_k$. Now, if $\lambda<\gamma$ there is set of power $\aleph_n$, composed of elements of the sequence (2) and contained between λ and γ. It follows that, if $\alpha<\omega_n$, each neighborhood of γ contains at least one element of G_{ξ_α}. As the sequence $G_{\xi_1}, G_{\xi_2}, \cdots, G_{\xi_\alpha}, \cdots$ $(\alpha<\omega_n)$ is confinal with $G_1, G_2, \cdots, G_\xi, \cdots$, it follows at once that each neighborhood of γ contains an element of G_ξ. Hence γ belongs to G'_ξ for each $\xi<\omega_n$.

* Hausdorff, *Grundzüge der Mengenlehre*, p. 132, Leipzig, Veit, 1914.

Thus E is perfectly compact and Theorem I is, in this case, false. It may be proved by a similar argument that for any irregular aleph there exists a set E for which the Theorem I does not hold true.

3. *The Corollary of Theorem* I *holds True for any Set* E. Since Chittenden gave a proof of the corollary in case the power of E, call it $\aleph_\alpha$, is a regular aleph, I shall suppose that $\aleph_\alpha$ is irregular. It follows that there exists an ordinal $\beta<\alpha$ (of the second kind) and a transfinite sequence of increasing ordinals $\alpha_\xi<\alpha$ $(\xi<\beta)$ such that $\lim_{\xi<\beta}\alpha_\xi=\alpha$ and

$$\aleph_\alpha = \sum_{\xi<\beta} \aleph_{\alpha_\xi}. \tag{3}$$

Suppose E is a perfectly self-compact set that does not contain any nuclear point. Then there exists for each element p of E a neighborhood $V(p)$ such that the product $E \cdot V(p)$ is of power $< \aleph_\alpha$. Let

$$p_1, p_2, p_3, \cdots, p_\omega, \cdots, p_\kappa, \cdots \qquad (\lambda < \omega_\alpha) \tag{4}$$

be the sequence (of the least possible type) constituted of all the elements of E.

Let $\xi<\beta$; let T_ξ be the sum of all the sets $E \cdot V(p_\lambda)$ such that $E \cdot V(p_\lambda)$ is of power $\leqq \aleph_{\alpha_\xi}$ and $\lambda<\omega_{\alpha_\xi}$. Thus T_ξ is obtained by adding together $\leqq \aleph_{\alpha_\xi}$ sets each of power $\leqq\aleph_{\alpha_\xi}$. It follows that the power of T_ξ is $\leqq \aleph_{\alpha_\xi}^2=\aleph_{\alpha_\xi}<\aleph_\alpha$ (since $\xi<\beta$ implies $\alpha_\xi<\alpha$). As E is of power $\aleph_\alpha$ there must exist in E a point q_ξ that does not belong to T_ξ. Now as the numbers α_ξ $(\xi<\beta)$ are increasing, it follows from the definition of T_ξ that

$$T_\xi \subset T_\eta \quad \text{if} \quad \xi \leqq \eta < \beta. \tag{5}$$

Let S_ξ denote the set of points q_η such that $\xi\leqq\eta<\beta$. The sets S_ξ are decreasing. Now, E is perfectly self-compact. Hence the sets S_ξ or their derived sets S_ξ' have a common point q. As q belongs to E there is an ordinal λ such that $q=p_\lambda$. As $\lambda<\omega_\alpha$ and $\lim_{\xi<\beta}\alpha_\xi=\alpha$, $\lim_{\xi<\beta}\omega_{\alpha_\xi}=\omega_\alpha$, there exists an ordinal $\rho<\beta$ such that $\lambda<\omega_{\alpha_\rho}$. Now, since the

set $E \cdot V(p_\lambda)$ is of power $< \aleph_\alpha$ there exists an ordinal $\zeta < \beta$ such that $E \cdot V(p_\lambda)$ is of power $\leqq \aleph_{\alpha_\zeta}$. Let $\xi = \rho$ in case $\rho \geqq \zeta$ and $\xi = \zeta$ in case $\rho < \zeta$. Thus: $\xi < \beta$, $\lambda < \omega_{\alpha_\xi}$ and $E \cdot V(p_\lambda)$ is of power $\leqq \aleph_{\alpha_\xi}$. Hence p_λ belongs to T_ξ. It follows from the definition of the sequence $\{q_\eta\}$ and from (5) that q_η does not belong to T_ξ if $\eta \geqq \xi$. Hence q_η does not belong to $V(q)$ if $\eta \geqq \xi$, since the set $E \cdot V(q) = E \cdot V(p_\lambda)$ is contained in T_ξ. Therefore $S_\xi \cdot V(q) = 0$, contrary to the hypothesis that q is a common point either of all the sets S_ξ or of all the S'_ξ.

Consequently, E must contain a nuclear point.

4. *Theorem* II *holds True for any Set* E. This may be proved as follows. Let S be a monotonic sequence of subsets of E. It may be assumed (cf. § 2) that S is a well-ordered series of decreasing sets G_α and that the ordinal number of S is an initial number ω_γ. Let p_α be in G_α but not in $G_{\alpha+1}$. The set H of all p_α is of power $\aleph_\gamma$.

By hypothesis, H determines a nuclear point q. Hence each neighborhood V of q contains $\aleph_\gamma$ points of H. It follows that V contains at least one point of every G_α. Therefore q belongs to all the G'_α. Consequently E is perfectly compact.

5. *Conclusions.* Thus Theorem II holds true for any E, but Theorem I may be false if the power of E is an irregular aleph. If follows at once that although *the condition stated in Theorem* III *is sufficient, it is not necessary.*

Consequently *the condition stated in the Corollary of Theorem* III *is sufficient.* However, *it is not necessary.*

For, let K be the set of all ordinals $\xi < \omega_\omega$ and call neighborhood of ξ any set of all ordinals η such that $\lambda < \eta \leqq \xi$ (where $\lambda < \xi$). By an argument analogous to that of § 2, it is easy to see that every compact subset E of K is perfectly compact. Evidently there does not exist in K any nuclear point.

THE UNIVERSITY OF WARSAW

ON THE INTEGRO-DIFFERENTIAL EQUATION OF THE BÔCHER TYPE IN THREE-SPACE

BY G. E. RAYNOR

1. *Introduction.* Bôcher has shown* that if a function $f(x, y)$ is continuous and has continuous first partial derivatives in a region R and satisfies the condition

$$\int_C \frac{\partial f}{\partial n}\, ds = 0$$

for every circle C lying entirely in R, then $f(x, y)$ is harmonic at each interior point of R. Bôcher treats only functions in two variables and by a method which cannot be directly extended to three-space.

It is the purpose of the present note to show, by a simple modification of the second part of Bôcher's argument, that this result may at once be extended to three-space, and also to investigate the nature of the function f if Bôcher's condition of continuity is somewhat weakened. We shall treat explicitly functions in three variables only, but it will easily be seen that with a slight modification the statements of Theorem II are applicable to two-space as well.

THEOREM I. *If a function $f(x, y, z)$ is continuous, and has continuous first partial derivatives in a connected finite region R, and is such that the surface integral $\int_S (\partial f/\partial n)ds$ vanishes when taken over every sphere S lying in R, then at each interior point of R, f is harmonic; that is, it satisfies Laplace's equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

at each interior point of R.

* PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 41, pp. 577–583.

In the preceding integral, as well as in what follows, the derivative $\partial f/\partial n$ is to be taken either toward the interior of S, or toward the exterior of S, throughout the region of integration. Let P be any interior point of R and consider two spheres S_1 and S_2 of radii r_1 and $r_2 < r_1$ with centers at P. By hypothesis, we have

$$\int_S \frac{\partial f}{\partial n}\, ds = 0\,;$$

or, setting $ds = r^2 d\omega$, where $d\omega$ is the element of area on the unit sphere with center P,

$$\int_S \frac{\partial f}{\partial n}\, d\omega = 0.$$

It follows that

$$\int_{r_2}^{r_1} dr \int_S \frac{\partial f}{\partial n}\, d\omega = 0. \tag{1}$$

Because of the continuity of f and its derivatives the order of integrations in the above integral may be inverted and we have

$$\int_{S_1} f d\omega - \int_{S_2} f d\omega = 0. \tag{2}$$

Let $f(P)$ be the value of f at the point P. Then since f is continuous at P we obtain from (2) by letting r_2 approach zero,

$$f(P) = \frac{1}{4\pi} \int_{S_1} f d\omega = \frac{1}{4\pi r_1^2} \int_{S_1} f ds.$$

We thus see that our function f possesses the so-called *mean-value property*, that is, its value at the center of any sphere is the mean of its values on the surface of the sphere.

Consider now the function F which takes the same values as f on S_1 and which is harmonic interior to S_1. This function exists and can be expressed as a Poisson integral. It is well known that F also possesses the mean-value prop-

erty and hence so also does the difference $f-F$. But a continuous function having the mean value property in a closed region R must take its greatest and least values on the boundary of R. Since the difference $f-F$ is identically zero on S_1 it follows that it is zero everywhere within S_1 and hence f must be harmonic at P as was to be proved.

It is evident that the original hypothesis that

$$\int \frac{\partial f}{\partial n} ds = 0$$

about every sphere in R is unnecessarily broad. All that is needed in the above proof is that each point P may be surrounded by a region, no matter how small, which is such that the above integral vanishes when taken over every sphere lying entirely within it.

2. *A More General Theorem.* We shall now weaken the original condition of continuity on f and suppose that it is continuous at every interior point of R except possibly at a finite number of points $P_1, P_2, \cdots, P_i, \cdots, P_n$. We shall refer to these exceptional points in the sequel as the points P_i. Our other condition on f now takes the form "about each interior point of R there exists a region M which is such that in its interior $\int (\partial f/\partial n) ds$ evaluated over every sphere which lies in M and does not pass through one of the P_i is zero." It is sufficient that if M contains one of the exceptional points, it contains only one.

That f is harmonic at every interior point P of R other than the P_i follows readily. About each of the P_i as center draw a small sphere S_i which does not contain P. Then the region bounded by the S_i and the boundary of R is a region of the type considered in Theorem I from which it follows that f is harmonic at P. It thus remains only to consider the nature of f in the neighborhood of any one of the P_i.

In a paper presented to the Society, October 31, 1925, the writer has shown that if a function is harmonic at every point

in the deleted neighborhood of a point P it may be expressed in the form

$$c\frac{1}{r} + \Phi(x,y,z) + V(x,y,z)$$

in this neighborhood. In this expression c is a constant, r the distance from P to (x, y, z), V a function harmonic everywhere in the neighborhood of P as well as at P itself and Φ a function harmonic in the deleted neighborhood and such that it is either identically zero or else there exist modes of approach to P for which Φ will tend toward plus infinity and also modes of approach for which it will tend toward minus infinity; Φ also possesses the property that its integral over the surface of any sphere with P as center vanishes.

Consider now two spheres S_1 and S_2 with center P and radii r_1 and $r_2 < r_1$. Apply Green's formula to the functions Φ and $1/r - 1/r_1$ for the region bounded by S_1 and S_2 and we have

$$\int_{S_1 S_2} \left\{ \left(\frac{1}{r} - \frac{1}{r_1}\right)\frac{\partial \Phi}{\partial n} - \frac{\partial}{\partial n}\left(\frac{1}{r} - \frac{1}{r_1}\right)\Phi \right\} ds = 0,$$

where the normal derivatives are taken toward the interior of the region $S_1 S_2$. Remembering that the integral of Φ over any sphere with center P is zero and since $1/r - 1/r_1$ is zero on S_1 and constant on S_2 we have from the above equation

$$\int_{S_2} \frac{\partial \Phi}{\partial n} ds = 0. \tag{3}$$

Since S_2 is any sphere interior to S_1 it follows that the integral of the normal derivative of Φ over any sphere with center P and radius less than r_1 vanishes. The same result could of course be obtained from the property $\int_S \Phi ds = 0$ by considering the continuity of Φ and using the theorem concerning the differentiation of a definite integral. In (3) because of the continuity of the first partial derivatives of Φ in the deleted neighborhood of P we may take the normal derivative *either* toward the interior or toward the

exterior normal of S_2 throughout the region of integration. Hence if the function

$$f = c\frac{1}{r} + \Phi + V$$

is to be such that the integral of its normal derivative vanishes when taken over spheres in the neighborhood of P, the constant c must be zero. Conversely we have shown by the above argument that if c is zero $\int (\partial f/\partial n)ds$ will vanish when taken over every sphere with center P and of radius $r < r_1$. We may now state the following theorem.

THEOREM II. *Every function which satisfies the conditions of § 2 in a region R is harmonic at every interior point of R except possibly at the points P_i. In the neighborhood of each P_i, f is of the form $\Phi + V$. If $\Phi \equiv 0$, in the neighborhood of any P_i, P_i is at most a removable discontinuity. If $\Phi \not\equiv 0$, f will be harmonic in the deleted neighborhood of P and will be such that for certain modes of approach to P it will tend toward plus infinity and for other modes to minus infinity.*

It may be remarked in closing that although we have supposed $\int (\partial f/\partial n)ds$ to vanish only when taken over sufficiently small spheres with P_i as center it is now easy to prove that it will vanish when taken over any regular surface S in R which does not pass through one of the P_i. We need merely to surround each P_i in S by a sphere S_i lying entirely in S and use the fact that

$$\int \frac{\partial f}{\partial n} ds = 0 \tag{4}$$

where the integral is taken over S and the spheres S_i. Then $\int_S (\partial f/\partial n)ds$ will vanish since the portion of (4) due to the S_i vanishes.

WESLEYAN UNIVERSITY

TWO-WAY CONTINUOUS CURVES*

BY G. T. WHYBURN

A continuous curve M will be said to be a two-way continuous curve, or to be "two-way continuous," provided it is true that between every two points of M there exist in M at least two arcs neither of which is a subset of the other. A point P of a continuum M is a cut point of M provided it is true that the point set $M-P$ is not connected. Every point of a continuum M which is not a cut point of M will be called a non-cut point of M.

In a paper *Concerning continua in the plane*,† among other results, I have established the following theorems which will be used in the proofs given in this paper.

I. *If K denotes the set of all the cut points of a continuum M, then every bounded, closed, and connected subset of K is a continuous curve which contains no simple closed curve.*

II. *Every cut point of the boundary of a complementary domain of a bounded continuum M is a cut point also of M.*

III. *If K, H, and N, respectively, denote the set of all the cut points, end points,‡ and simple closed curves of a continuous curve M, then $K+H+N=M$.*

IV. *If N denotes the point set consisting of all the simple closed curves contained in a continuous curve M, then every connected subset of $M-N$ is arcwise connected.*

These results will be referred to by number as here listed. We shall now prove the following additional theorems.

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* Presented to the Society, May 1, 1926.

† Recently submitted for publication in the TRANSACTIONS OF THIS SOCIETY.

‡ For a definition of this term see R. L. Wilder, *Concerning continuous curves*, FUNDAMENTA MATHEMATICAE, vol. 7 (1925), p. 358.

THEOREM 1. *In order that a continuous curve M should be two-way continuous it is necessary and sufficient that every simple continuous arc of M should contain a subarc which belongs to some single simple closed curve of M.*

THEOREM 2. *In order that a continuous curve M should be two-way continuous it is necessary and sufficient that every arc of M should contain a non-cut point of M.*

Proof. The condition is sufficient. Let A and B denote any two points of a continuous curve M which satisfies the condition. The curve M contains one arc t from A to B. And from our hypothesis it follows that t contains an interior point O which is a non-cut point of M. It follows from a theorem of R. L. Moore's* that $M-O$ contains an arc s from A to B. Since s does not contain the point O of t, it follows that $t \neq s$, and therefore, that M is two-way continuous.

The condition is also necessary. Let t denote any definite arc of a two-way continuous curve M. By Theorem 1, t contains a subarc s which belongs to some simple closed curve J of M. It is a consequence of a theorem of R. L. Moore's† that J contains not more than a countable number of cut points of M. Since s belongs to J and contains uncountably many points altogether, it follows that s, and hence also t, must contain at least one non-cut point of M.

THEOREM 3. *In order that a continuous curve M should be two-way continuous it is necessary and sufficient that the set K of all the cut points of M should contain no continuum.*

Proof. That the condition is sufficient is almost a direct consequence of Theorem 2. For, since by hypothesis K can contain no continuum, therefore it can contain no arc.

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* *Concerning continuous curves in the plane*, MATHEMATISCHE ZEITSCHRIFT, vol. 15 (1922), pp. 254–260, Theorem 1.

† *Concerning the cut points of continuous curves and of other closed and connected point sets*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 9 (1923), pp. 101–106, Theorem B*.

Hence, every arc of M must contain a non-cut point of M, and by Theorem 2, M is two-way continuous. The condition is also necessary. For suppose the set K of all the cut points of a two-way continuous curve M contains a continuum H. Then by (I), H is a continuous curve. Hence, H contains at least one arc t. But by Theorem 2, t must contain at least one non-cut point of M. Thus the supposition that K contains a continuum leads to a contradiction.

THEOREM 4. *The boundary of every complementary domain of a two-way continuous curve is itself two-way continuous.*

Proof. Let M denote the boundary of a complementary domain of a two-way continuous curve K. Then M is a continuous curve.* Suppose, contrary to this theorem, that M is not two-way continuous. Then from Theorem 3 it follows that M must contain a continuum H every point of which is a cut point of M. But by (II), every cut point of M is a cut point also of K. And since K is two-way continuous, by Theorem 3, not every point of H can be a cut point of K. Thus the supposition that M is not two-way continuous leads to a contradiction.

THEOREM 5. *If N denotes the point set consisting of all the simple closed curves contained in a two-way continuous curve M, then $M-N$ is totally disconnected.*

Proof. Suppose $M-N$ contains a connected set L consisting of more than one point. Then from (III) and (IV) it readily follows that L contains an arc t every point of which is a cut point of M. But this is contrary to Theorem 2. It follows that $M-N$ is totally disconnected.

THEOREM 6. *The boundary M of a complementary domain of a two-way continuous curve is the sum of two mutually exclusive point sets N and H, where N is the sum of a countable*

* Cf. Miss Torhorst, *Über den Rand der einfach zusammenhängenden ebenen Gebiete*, MATHEMATISCHE ZEITSCHRIFT, vol. 9 (1921), p. 64 (73).

number of simple closed curves no two of which have more than one point in common, and H is a totally disconnected set of points every one of which is a limit point of N and is either a cut point or an end point of M.

Proof. By Theorem 4, M is a two-way continuous curve. Let G denote the collection of all the simple closed curves contained in M. R. L. Wilder* has shown that G is countable and that no two curves of G have more than one point in common. Let N denote the point set obtained by adding together all the curves of the collection G. Then let H denote the point set $M-N$. Since M is two-way continuous, it readily follows that every point of H is a limit point of N. By Theorem 5, H is totally disconnected, and by (III), every point of H is either a cut point or an end point of M. Hence, the sets N and H satisfy all the conditions of Theorem 6.

THEOREM 7. *In order that the boundary M of a complementary domain D of a continuous curve should be two-way continuous it is necessary and sufficient that M should contain a point set K such that* (1) $D+K$ *is uniformly connected im kleinen, and* (2) *every arc, if there be any, which* K' (*K plus all the limit points of K*) *contains, contains a non-cut point of M.*

Proof. The condition is necessary. For let $K=M$. Clearly $D+K$ is uniformly connected im kleinen. And since K' is two-way continuous, it follows by Theorem 2 that every arc of K' contains a non-cut point of M. The condition is also sufficient. Let M denote the boundary of a complementary domain D of a continuous curve, and suppose that M contains a point set K satisfying conditions (1) and (2) in the statement of Theorem 7. Let A and B denote any two points of M. Now M contains one arc t from A to B. Either t is a subset of K' or it is not. If t is a subset of K', then by hypothesis t contains an interior

* Loc. cit., Theorem 4.

point O which is not a cut point of M. Then by a theorem of. R. L. Moore's,* $M-O$ contains an arc from A to B which does not contain O, and which, therefore, is not a subset of t. Now if t is not a subset of K', then since K' is closed, it readily follows that t contains an arc s which contains no point of K'. Let X and Y denote the end points and O an interior point of s. Let C be a circle having O as center and not enclosing any point of K. Within C and on s there exist points E, U, W, and G in the order X, E, U, O, W, G, Y. And within C there exist arcs EFG and UVW having only their end points in common with s and such that if D_1 and D_2 denote the interiors of the closed curves $UVWOU$ and $EUOWGFE$ respectively, then D_1 and D_2 are mutually exclusive domains each of which lies within C. Now since $D+K$ is uniformly connected im kleinen, and C encloses no point of K, it readily follows that not both D_1 and D_2 can contain a subset of D which has O for a limit point. Hence, either D_1 or D_2 must contain a segment QST of an arc QST which has its end points on s in the order X, Q, O, T, Y and such that if R denotes the interior of the closed curve $QOTSQ$, then R contains no point whatever of $D+M$. Hence, R lies wholly in some complementary domain G of $D+M$. It follows from a theorem of R. L. Moore's† that the boundary J of G is a simple closed curve. The curve J contains the arc QOT of t. It follows that M contains an arc from A to B which does not contain the point O of t, and which, therefore, is not a subset of t. Hence, in any case, M contains two arcs from A to B neither of which is a subset of the other, and therefore M is two-way continuous.

THE UNIVERSITY OF TEXAS

* *Concerning continuous curves in the plane*, loc. cit.

† *Concerning continuous curves in the plane*, loc. cit., Theorem 4.

ON SOME THEOREMS OF BÔCHER CONCERNING ISOLATED SINGULAR POINTS OF HARMONIC FUNCTIONS*

BY O. D. KELLOGG

In a note entitled *Deux théorèmes élémentaires sur les singularités des fonctions harmoniques*,† Picard gave proofs of the following theorems.

I. *If $u(x, y)$ is single-valued and harmonic in the neighborhood of a point A, except at A, and is bounded, then $u(x, y)$ becomes harmonic at A also when its definition at this point is properly adjusted.*

II. *If $u(x, y)$ is single-valued and harmonic in the neighborhood of A, except at A, and becomes positively infinite at A, then $u(x, y) - h \log (1/r)$ is harmonic at A, where h is a properly chosen constant, and r is the distance of (x, y) from A.*

As credit for these results was not assigned in the note cited, the author must have been unaware of a paper by Bôcher,‡ in which were proved the above theorems, and several others, not only for the plane, but for n dimensions, and for other differential equations of the elliptic type as well as Laplace's.

In addition to calling attention to the above matter of authorship, the object of this paper is to give an elementary proof of a theorem from which follow at once the results given by Bôcher, as far as solutions of Laplace's equation

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* Presented to the Society, September 9, 1926. See the concluding footnote.

† COMPTES RENDUS, vol. 176 (1923), pp. 933–35.

‡ *Singular points of functions which satisfy partial differential equations of the elliptic type*, this BULLETIN, vol. 9 (1903), pp. 455–465.

are concerned. We shall reason in three dimensions; the same argument is available for the corresponding theorems in any space of two or more dimensions.

We first derive a result of which a special case has been given by Lebesgue.* Here, and in the following, we shall use N to denote a neighborhood of the point O from which O has been removed. The harmonic functions are all assumed to be one-valued.

THEOREM I. *Let $U(P)$ be harmonic in N, and such that there exists a positive integer, m, for which*

$$\lim_{r\to\infty} r^m U(P) = 0,$$

uniformly. Then in N,

$$(1)\quad U(P) = V(P) + \frac{S_0(\phi,\theta)}{r} + \frac{S_1(\phi,\theta)}{r^2} + \cdots + \frac{S_{m-2}(\phi,\theta)}{r^{m-1}},$$

where $r^i S_i(\phi, \theta)$ is the usual spherical harmonic of order i, and $V(P)$ is harmonic throughout a neighborhood of O.

The existence of the infinite development for $U(P)$ corresponding to (1) is well known; also the uniform convergence of this development on all spheres about O of sufficiently small positive radius:

$$(2)\quad U(P) = \Theta\delta(r)/r^m = V(P) + \frac{S_0(\phi,\theta)}{r} + \frac{S_1(\phi,\theta)}{r^2} + \cdots.$$

Our hypothesis is contained in this formula if it is understood that $|\Theta| \leqq 1$, and that $\delta(r)\to 0$ with r.

The coordinates of P are (r, ϕ, θ). Let both members of the equation (2) be multiplied by $P_n(\cos\gamma)\sin\theta$, and integrated from 0 to π with respect to θ, and from 0 to

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* *Sur les singularités des fonctions harmoniques*, COMPTES RENDUS, vol. 176 (1923), pp. 1270–71.

2π with respect to ϕ, where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cdot\cos(\phi-\phi')$. The result is

$$(3)\quad \frac{\delta(r)}{r^m}\int\int\Theta(r,\phi,\theta)P_n(\cos\gamma)\sin\theta d\theta d\phi = \int\int V(r,\phi,\theta)P_n(\cos\gamma)\sin\theta d\theta d\phi + \frac{4\pi}{2n+1}\cdot\frac{S_n(\phi',\theta')}{r^{n+1}}.$$

If, now, we multiply the terms of this equation by r^{n+1}, and allow r to approach 0, we see that if $n-m+1\geqq 0$, $n\geqq 0$, $S_n(\phi',\theta')\equiv 0$. Thus the equation (1) is established.

THEOREM II. *Let $U(P)$ be harmonic in N, and be either bounded above, or bounded below. Then in N*

$$U(P) = c/r + V(P),$$

*where $V(P)$ is harmonic throughout a neighborhood of O.**

The proof is based on inequalities of Harnack† for positive harmonic functions, which, in three-dimensional space, take the form

$$(4)\quad \frac{(R-r)R}{(R+r)^2}U(C) \leqq U(P) \leqq \frac{(R+r)R}{(R-r)^2}U(C)$$

where $U(P)$ is harmonic and not negative throughout the sphere of radius R and with C as center, and where $r=CP$.

We may reduce our problem to one in which $U(P)\geqq 0$ in N by a change of sign and the addition of a constant, if necessary. Now let a be a positive number such that a sphere of radius $2a$ about O contains, with the exception of O, only points of N, and let M be the maximum of $U(P)$ on a sphere of radius a about O. Let θ be a constant between 0 and 1. We apply the second inequality (4) to $U(P)$,C

* This theorem is a direct consequence of the theorems of Bôcher, V and II, loc. cit., pp. 461 and 456.

† *Theorie des logarithmischen Potentials*, Leipzig, 1887, p. 62.

being on the sphere of radius a about O, R being θa, and r being θR, or $\theta^2 a$. The result is

$$U(P) \leqq \frac{1+\theta}{(1-\theta)^2} M, \quad \text{for} \quad a(1-\theta^2) \leqq OP \leqq a.$$

With this bound for $U(P)$ on the sphere of radius $a(1-\theta^2)$ about O, we again apply the inequality of (4), taking C on this sphere, and replacing a by $a(1-\theta^2)$. After a repetition of this process, we obtain the general result

$$U(P) \leqq \left[\frac{1+\theta}{(1-\theta)^2}\right]^n M, \quad \text{for} \quad a(1-\theta^2)^n \leqq OP \leqq a(1-\theta^2)^{n-1}.$$

Let us now replace n by a continuous variable, s, and write $r = a(1-\theta^2)^s$. Our result then yields the inequality

$$U(P) \leqq \left[\frac{1+\theta}{(1-\theta)^2}\right]^{s+1} M, \quad \text{for} \quad n-1 \leqq s \leqq n, \tag{5}$$

since the bound for $U(P)$ is an increasing function of s. But as this bound is independent of n, (5) holds for all $s \geqq 1$. If, now, we replace s by its value in terms of r, we arrive at the result

$$U(P) \leqq \frac{1+\theta}{(1-\theta)^2} \cdot \frac{M}{a} \cdot r^p,$$

where

$$p = \frac{\log(1+\theta) - 2\log(1-\theta)}{\log(1+\theta) + \log(1-\theta)}.$$

As $\theta \to 1$, p approaches -2 from below. It follows that $U(P)$ is subject to an inequality of the type $U(P) \leqq Mr^{-2-\eta}$, where η may be any fixed positive number.

Such a function, as we have seen in Theorem I, is of the form

$$U(P) = V(P) + c/r + S_1(\phi, \theta)/r^2.$$

But if $S_1(\phi, \theta)$ were not identically 0, it would have both positive and negative values, and hence for small enough r, $U(P)$ would have to assume arbitrarily great positive

and negative values. But this contradicts our hypothesis. Therefore $S_1(\phi, \theta) \equiv 0$, and $U(P)$ has the stated form.

We close with some consequences of the above two theorems. In the first place, the theorem on isolated singularities of bounded harmonic functions (the first one quoted at the outset) is included in Theorem I.*

The second theorem quoted is a consequence of Theorem II,* for a function, harmonic in N, which becomes positively infinite at O must be positive in some neighborhood of O (with O omitted).

Finally, we may also infer the theorem of Bôcher†: *If a function $U(P)$ of two or more variables, harmonic in N, becomes infinite for some but not for all modes of approach of P to 0, then it can be made to become both positively and negatively infinite (and therefore also to take on every real value an infinite number of times) by suitable choice of the mode of approach of P to 0.* For if $U(P)$ did not become both positively and negatively infinite in the neighborhood of 0, it would, by Theorem II, become positively infinite for every mode of approach, since the fact that it becomes infinite for some mode of approach assures us that c is not 0 in that theorem.‡

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* It being understood that the wording is suitably altered to meet the two-dimensional case.

† Loc. cit., footnote 3, p. 664.

‡ At the time of writing this note, the author was unaware of a paper on the same subject read before the Society October 31, 1925, by G. E. Raynor, and published in the last number of the BULLETIN. An earlier knowledge of Raynor's paper would have resulted in modifications of the above, but under the circumstances it seemed best to publish it unaltered and this has been done.

Although the papers are written, one for two and the other for three dimensions, the methods used are all available for any space $n \geqq 2$. Theorem II of the present note is implied by Raynor's IV (Böcher's V, loc. cit.), while Theorem II, above, is independent, and involves an additional hypothesis.

THE TRANSVERSALITY RELATIVE TO A SURFACE OF $\int F(x, y, z, y', z')dx = \text{MINIMUM}$*

BY JESSE DOUGLAS†

1. *Introduction and Statement of Theorem.* Consider, for illustration, a surface Σ, on which A and B are any two points. Then, of all curves in space which join A and B, the straight line segment AB is the shortest; of all curves on Σ joining A and B, the geodesic AB is the shortest.

The generalization from the length integral

$$s = \int (1 + y'^2 + z'^2)^{1/2} dx \tag{1}$$

to the general integral of first order

$$J = \int F(x, y, z, y', z')dx \tag{2}$$

is obvious. Thus we may speak on the one hand of the unrestricted extremals of J relative to space, ∞^4 in number, and, on the other, of the extremals of J relative to a given surface Σ, ∞^2 in number.

The idea of *transversality* may likewise be defined relative to a given surface as well as for space. Let us review the well-known definitions and facts in this connection. The space transversality T belonging to an integral J is essentially a correspondence between lineal elements and surface elements (of the first order) characterized by the following two properties: (1) a lineal element and its corresponding surface element have the same base point, (2) if, taking an arbitrary base surface S, we construct the ∞^2 extremals of J which meet S transversally, then lay off along each extremal, starting at S, an arc over which the integral J has a fixed value, the locus of the end points of these arcs is a surface trans-

* Presented to the Society, October 30, 1926.

† National Research Fellow in Mathematics.

versal to the congruence of extremals. By varying the fixed value continuously ∞^1 such transversal surfaces result.

Analytically, if a lineal element be fixed by coordinates x, y, z, y', z' in the usual way, the transverse surface element has the same base point x, y, z, and the equation of its (infinitesimal piece of a) plane is

$$(3) \qquad (F - y'F_{y'} - z'F_{z'})\delta x + F_{y'}\delta y + F_{z'}\delta z = 0,$$

where $\delta x, \delta y, \delta z$ denote coordinates relative to x, y, z.

The surface S or any of the other transversal surfaces of the congruence of extremals may degenerate into a curve or a point—in fact, the appropriate concept to use here is not that of surface but Lie's concept of *union* of surface elements. The simplest representation of a transversality is obtained by allowing S to be a point—the extremals transversal to a point are simply those which issue from it. In the illustrative case of the length integral (1) the transversal surfaces are then concentric spheres, from which it appears that for the length integral transversality is the same as orthogonality*—a fact which is expressed analytically by the circumstance that the equation (3) formed for (1) reduces to $\delta x + y'\delta y + z'\delta z = 0$.

How is the transversality of J relative to an arbitrary surface Σ defined? Let us construct the pencil of extremals of J relative to Σ which radiate from a fixed point 0 of Σ. If we proceed along each extremal, starting from 0, until we have run off a fixed value of J, the locus of points arrived at is a curve Γ, which we say meets transversally the extremals radiating from 0. More definitely, if C is one of the extremals through 0, intersecting Γ at point p, the lineal element λ of Γ at p is termed transversal to the lineal element l of C at the same point. By varying 0, there is assigned to each lineal element l of Σ a transverse lineal element λ.

* The most general integral of the first order for which transversality is orthogonality is $\int \nu(x, y, z)ds$, where ν is any function of its three arguments.

If C is the extremal containing the element l, any point of C may be used as point 0 to define the transverse element λ; we always get the same λ.

The correspondence so defined between l and λ we call the *transversality* of J *relative to* Σ, and denote by τ; it is of course to be carefully distinguished from the space transversality T of J. The analogy between τ and T is completed by the theorem that the extremals of J relative to Σ which start transversally from any base curve have ∞^1 transversal curves, between any two of which arcs over which J has the same value are intercepted on all these extremals.

Now, in the theory of surfaces there is a classic theorem of Gauss which may be stated as follows.

Any two orthogonal trajectories of a system of ∞^1 *geodesics of a surface intercept the same length on all these geodesics.**

In our terminology the theorem may be stated as follows:

The transversality τ *of the length integral relative to an arbitrary surface* Σ *is the orthogonality of lineal elements on* Σ.

The purpose of the present paper is to prove the following more general theorem, which includes that of Gauss as a special case.

THEOREM. *The transversality* τ *of any integral* J *relative to an arbitrary surface* Σ *is the section by* Σ *of the space transversality* T *of* J.

By the section of T by Σ we mean, naturally, the following: Let l be any lineal element of Σ, and let the surface element σ correspond to l by T; then the lineal element λ corresponding to l by τ is the intersection of (the tangent plane of) Σ with σ.

2. *Proof.* Let Σ be represented by the equation

$$x + \phi(y,z) = 0. \tag{4}$$

* G. Scheffers, *Einführung in die Theorie der Flächen*, second edition, 1913, p. 500.

Let

$$y = y(x), \quad z = z(x) \tag{5}$$

define a curve C on Σ which joins any two fixed points 0 and 1 of that surface. Then the infinitely near curve $\overline{C}$

$$y = y(x) + \epsilon\eta(x), \qquad z = z(x) + \epsilon\zeta(x) \tag{6}$$

may be considered as lying on Σ when and only when

$$p\eta(x) + q\zeta(x) = 0 \tag{7}$$

where $p = \phi_y(y, z)$, $q = \phi_z(y, z)$ and y, z are to be replaced by the functions of x appearing in (5); (7) then to hold identically in x.

The conditions that $\overline{C}$ pass through 0 and 1 are

$$\eta(x_0) = 0, \ \zeta(x_0) = 0 \ ; \ \eta(x_1) = 0, \ \zeta(x_1) = 0. \tag{8}$$

Supposing (7) and (8) verified, the variation of J in passing from C to $\overline{C}$ is, by means of the usual integration by parts,

$$\delta J = \int_0^1 \{A(F)\eta + B(F)\zeta\}\, dx \tag{9}$$

where

$$A(F) = F_y - \frac{d}{dx}F_{y'}, \ B(F) = F_z - \frac{d}{dx}F_{z'}, \tag{10}$$

d/dx denoting total differentiation with respect to the abscissa of C. In order that C be an extremal of J relative to Σ, the necessary and sufficient condition is

$$A(F)\eta + B(F)\zeta = 0 \tag{11}$$

for all η, ζ satisfying (7), or $qA(F) - pB(F) = 0$ where all the symbols involved are to be reduced to functions of x, abscissa along C.

Suppose now 2 to denote any point on Σ infinitely near to 1, and let C' be the extremal relative to Σ which joins 0 to 2. Its equations have the form (6) with (7) verified, and $\eta(x_0) = 0$, $\zeta(x_0) = 0$, but instead of the second pair of equations (8) we have, expressive of the fact that C' contains 2,

(12) $$\delta y = y_1'\delta x + \epsilon\eta(x_1), \quad \delta z = z_1'\delta x + \epsilon\zeta(x_1),$$

where δx, δy, δz denote the coordinates of 2 relative to 1 and x_1, y_1, z_1, y_1', z_1' are the coordinates of the lineal element of C at 1.

These equations then follow:

$$J_{01} = \int_{x_0}^{x_1} F(x, y, z, y', z')dx,$$

$$J_{02} = \int_{x_0}^{x_1+\delta x} \{F + \epsilon(F_y\eta + F_z\zeta + F_{y'}\eta' + F_{z'}\zeta')\}dx$$

$$= \int_{x_0}^{x_1} \{F + \epsilon(F_y\eta + F_z\zeta + F_{y'}\eta' + F_{z'}\zeta')\}dx + F(x_1, y_1, z_1, y_1', z_1')\delta x,$$

where we neglect infinitesimals of higher than the first order. By subtraction we find

$$\delta J = J_{02} - J_{01}$$

$$= \epsilon\int_{x_0}^{x_1}(F_y\eta + F_z\zeta + F_{y'}\eta' + F_{z'}\zeta')dx + F(x_1, y_1, z_1, y_1', z_1')\delta x$$

$$= \epsilon\{F_{y'}(1)\eta(x_1) + F_{z'}(1)\zeta(x_1)\} + \epsilon\int_{x_0}^{x_1}\{A(F)\eta + B(F)\zeta\}dx + F(1)\delta x.$$

The integral vanishes by (11), since η, ζ satisfy (7) because C' lies on Σ. Using (12), we have after dropping the index 1,

(13) $$\delta J = (F - y'F_{y'} - z'F_{z'})\delta x + F_{y'}\delta y + F_{z'}\delta z.$$

For transversality, $\delta J = 0$, giving

(14) $$(F - y'F_{y'} - z'F_{z'})\delta x + F_{y'}\delta y + F_{z'}\delta z = 0.$$

Besides (14), δx, δy, δz are restricted to satisfy

(15) $$\delta x + p\delta y + q\delta z = 0,$$

which is the equation of the element of Σ in the immediate neighborhood of 1 (tangent plane). The direction $\delta x : \delta y : \delta z$ transverse to $1 : y' : z'$ at 1 is to be found by combining (14) with (15).

Now (14) is identical with (3) defining the space transversality of J. Hence our theorem.

The theorem is easily extended to an m space immersed in an n space. Since a Riemann space of m dimensions can always be immersed in a flat space of at most $\frac{1}{2}m(m+1)$ dimensions, we have an immediate proof of the Gauss theorem for a Riemann space of any number of dimensions. For it is obvious that the transversality of the length integral in a flat n-space is the orthogonality of lineal elements to $(n-1)$-elements with the same base point, and evidently the section of this transversality by any m-spread contained in the n-flat is the orthogonality of lineal to $(m-1)$-elements in the m-spread.

PRINCETON UNIVERSITY

ON THE EXTENSION OF A METHOD OF BRIOT AND BOUQUET FOR THE REDUCTION OF SINGULAR POINTS*

BY B. O. KOOPMAN

In a classical memoir,† Briot and Bouquet gave a method by means of which the differential equation

$$\frac{dx}{X(x,y)} = \frac{dy}{Y(x,y)}$$

could be reduced to a simple standard form in the neighborhood of an analytic singular point, i. e., a point at which $X(x, y)$ and $Y(x, y)$ are analytic, but vanish simultaneously. Although the method fails to be directly applicable to certain special cases, it has shown itself to be of sufficient

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* Presented to the Society September 9, 1926.

† JOURNAL DE L'ÉCOLE POLYTECHNIQUE, vol. 21, p. 161. See also Picard, *Traité d'Analyse*, Paris, Gauthier-Villars, 1908, vol. 3, p. 34.

value* to warrant its extension to systems of higher order, an extension which forms the subject of the present note.

For clarity of exposition, we shall deal with the system of the second order

$$\frac{dx}{X(x,y,z)} = \frac{dy}{Y(x,y,z)} = \frac{dz}{Z(x,y,z)} \tag{1}$$

for which the origin is an analytic singular point:

$$\begin{cases} X = \sum_{\alpha',\beta',\gamma'}^{1-\infty} A'_{\alpha'\beta'\gamma'} x^{\alpha'} y^{\beta'} z^{\gamma'}, \\ Y = \sum_{\alpha'',\beta'',\gamma''}^{1-\infty} A''_{\alpha''\beta''\gamma''} x^{\alpha''} y^{\beta''} z^{\gamma''}, \\ Z = \sum_{\alpha''',\beta''',\gamma'''}^{1-\infty} A'''_{\alpha'''\beta'''\gamma'''} x^{\alpha'''} y^{\beta'''} z^{\gamma'''}, \\ \alpha^{(i)} + \beta^{(i)} + \gamma^{(i)} \geqq 1. \end{cases} \tag{2}$$

We wish to find the integral curves which approach the origin, and have there the form

$$x = ut^p, \qquad y = vt^q, \qquad z = wt^r. \tag{3}$$

Here u, v, w are functions of the parameter t, distinct from zero when $t=0$, and p, q, r, positive integers. When these expressions are substituted in equation (1), we get

$$\begin{aligned} &\frac{updt + tdu}{\sum_{\alpha',\beta',\gamma'} A'_{\alpha'\beta'\gamma'} u^{\alpha'} v^{\beta'} w^{\gamma'} t^{(\alpha'-1)p+\beta' q+\gamma' r+1}} \\ &= \frac{vqdt + tdv}{\sum_{\alpha'',\beta'',\gamma''} A''_{\alpha''\beta''\gamma''} u^{\alpha''} v^{\beta''} w^{\gamma''} t^{\alpha'' p+(\beta''-1)q+\gamma'' r+1}} \\ &= \frac{wrdt + tdw}{\sum_{\alpha''',\beta''',\gamma'''} A'''_{\alpha'''\beta'''\gamma'''} u^{\alpha'''} v^{\beta'''} w^{\gamma'''} t^{\alpha''' p+\beta''' q+(\gamma'''-1)r+1}}. \end{aligned} \tag{4}$$

* See J. Malmquist, *Sur les points singuliers des équations différentielles*, ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 15 (1921).

Suppose that p, q, r have been so chosen that the terms of lowest degree in t in these denominators are of the same degree, δ. Then we must have, for a certain set, $\alpha_0', \beta_0', \cdots, \gamma_0'''$, of the exponents,

$$(5) \qquad \begin{cases} (\alpha_0' - 1)p + \beta_0' q + \gamma_0' r = \delta - 1, \\ \alpha_0'' p + (\beta_0'' - 1)q + \gamma_0'' r = \delta - 1, \\ \alpha_0''' p + \beta_0''' q + (\gamma_0''' - 1)r = \delta - 1. \end{cases}$$

Our first problem is, then, to determine values of α', $\beta', \cdots, \gamma'''$ in equations (2) such that there exist positive integers, p, q, r which have the following properties: (i) p, q, r satisfy (5). (ii) If

$$A'_{\alpha_1'\beta_1\gamma_1'}, \quad A''_{\alpha_1''\beta_1''\gamma_1''} \quad A'''_{\alpha_1'''\beta_1'''\gamma_1'''}$$

are any other non-vanishing coefficients of terms of equation (2), and if $\alpha_0', \beta_0', \cdots, \gamma_0'''$ are replaced by $\alpha_1', \beta_1', \cdots, \gamma_1'''$, respectively, in (5), the left-hand members take on values which are not less than $\delta - 1$.

To find these quantities, we proceed geometrically. In the 3-dimensional space of rectangular coordinates (ξ, η, ζ), plot the following three classes of points:

(I) The points $(\alpha' - 1, \beta', \gamma')$, corresponding to all the non-vanishing $A'_{\alpha'\beta'\gamma'}$'s present in (2).

(II) The points $(\alpha'', \beta'' - 1, \gamma'')$, corresponding to the $A''_{\alpha''\beta''\gamma''}$'s.

(III) The points $(\alpha''', \beta''', \gamma''' - 1)$, corresponding to the $A'''_{\alpha'''\beta'''\gamma'''}$'s.

All these points lie within or upon the solid angle formed by the three quarter-planes

$$\xi = -1, \quad \eta \geqq -1, \quad \zeta \geqq -1;$$
$$\xi \geqq -1, \quad \eta = -1, \quad \zeta \geqq -1;$$
$$\xi \geqq -1, \quad \eta \geqq -1, \quad \zeta = -1.$$

Unless X, Y, Z are polynomials, the points will spread out to infinity. Let Π be a plane which cuts all three edges of

the solid angle, and which determines with the faces a tetrahedron not containing any of the points (I), (II), or (III) in its interior. Suppose that there lies on II at least one point $(\alpha_0' - 1, \beta_0', \gamma_0')$, one point $(\alpha_0'', \beta_0'' - 1, \gamma_0'')$, and one point $(\alpha_0''', \beta_0''', \gamma_0''' - 1)$. Then it is clearly possible to take as direction components of the normal to II three relatively prime positive integers, which we will call p, q, r. The equation of II will then be

$$p\xi + q\eta + r\zeta = \rho. \tag{6}$$

I say that the $\alpha_0', \beta_0', \cdots, \gamma_0''', p, q, r$ determined in the above manner have the requisite properties (i), (ii). That they satisfy the first requirement is seen from the fact that $(\alpha_0' - 1, \beta_0', \gamma_0')$, etc., lie on II, (taking $\delta = \rho + 1$). As for the second requirement, let $\alpha_1', \beta_1', \cdots, \gamma_1'''$ be any set of values represented in (2). By construction, we know that the points $(\alpha_1' - 1, \beta_1', \gamma_1')$, etc., do not lie on the same side of II as the vertex $(-1, -1, -1)$. It follows, on applying the rudiments of analytic geometry to the equation (6) and the points in question, that

$$(\alpha_1' - 1)p + \beta_1' q + \gamma_1' r \geqq p = \delta - 1, \text{ etc.}$$

i. e., requirement (ii) is satisfied.

We are now ready to complete the reduction. Let u_0, v_0, w_0 be the values which u, v, w take on when t equals zero. Then, when the common factor t^δ is removed from equations (4) and t set equal to zero, they yield

$$\frac{u_0 p}{SA'_{\alpha'\beta'\gamma'} u_0^{\alpha'} v_0^{\beta'} w_0^{\gamma'}} = \frac{v_0 q}{SA''_{\alpha''\beta''\gamma''} u_0^{\alpha''} v_0^{\beta''} w_0^{\gamma''}} = \frac{w_0 r}{SA'''_{\alpha'''\beta'''\gamma'''} u_0^{\alpha'''} v_0^{\beta'''} w_0^{\gamma'''}}. \tag{7}$$

Here S denotes a summation extended over those values of $\alpha', \beta', \cdots, \gamma'''$ which satisfy equations (5). It will in general be possible to select arbitrarily one of the quantities u_0, v_0, w_0, when values of the others will be determined by equations (7).

Everything has been done so far on the assumption that a plane Π exists. We now add some further hypotheses, which are fulfilled in general:

(a) There exists a set of quantities u_0, v_0, w_0, none of which are zero, which satisfy equations (7).

(b) The solutions u_0, v_0, w_0 are simple roots of (7).

(c) None of the quantities $SA'_{\alpha'\beta'\gamma'} u_0^{\alpha'} v_0^{\beta'} w_0^{\gamma'}$, etc., are zero for the solutions u_0, v_0, w_0 in question.

Suppose that it is desired to regard x as the independent variable. Take $u=1$ in equations (3) (this amounts to replacing t by the new variable $u^{1/p}t$), and perform the same substitutions as before, obtaining equations (4) and (7), only with $u=u_0=1$, $du=0$. Now set $v=v_0+\sigma_1$, $w=w_0+\sigma_2$; our equations will reduce to

$$(8) \qquad \begin{cases} t\dfrac{d\sigma_1}{dt} = a_1 t + b_1\sigma_1 + c_1\sigma_2 + \phi_1(t,\sigma_1,\sigma_2), \\ t\dfrac{d\sigma_2}{dt} = a_2 t + b_2\sigma_1 + c_2\sigma_2 + \phi_2(t,\sigma_1,\sigma_2), \end{cases}$$

where the expansions of ϕ_1 and ϕ_2 start with terms of the second degree or higher in t, σ_1, σ_2.

We have thus performed the reduction, corresponding to that of Briot and Bouquet, for the general system of the second order. Its extension to the nth order is carried out in a precisely similar manner, with the aid of the simplest notions of hypergeometry. In case certain of our hypotheses fail to be realized, we can still obtain results from the method of reduction; it is useless to elaborate this point further here, as the methods can be worked out in analogy with those for the first order.*

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* See Picard, loc. cit.; Malmquist, loc. cit.

A CURIOUS IRREDUCIBLE CONTINUUM

BY W. A. WILSON

One of Janiszewski's theorems* is to the effect that, if ab is a bounded irreducible continuum and c is a point of the first species, there is a decomposition of ab into two continua, ac and cb, such that $ac \cdot cb = c$. The fact that the condition imposed on c is not a necessary one naturally led to the question as to whether an irreducible continuum not made up of indecomposable continua can always be decomposed into two continua having only one point in common. A vain endeavor to answer this question in the affirmative resulted in the following example to the contrary, which may be of interest to workers in this field.

Although the continuum is simple, it is difficult to describe without a figure and therefore the construction should be carried out through the first two stages.

The first stage in the construction is as follows. Take a unit square in the first quadrant and mark the following points: $a=(0, 0)$, $b=(1/4, 0)$, $c=(1/2, 0)$, $d=(3/4, 0)$, $e=(1, 0)$, $f=(1/2, 1/4)$, $g=(3/4, 1/4)$, $h=(1/4, 3/4)$, $i=(1/2, 3/4)$, $j=(0, 1)$, $k=(1/4, 1)$, $l=(1/2, 1)$, $m=(3/4, 1)$, $n=(1, 1)$. Draw the straight lines bk, cf, fh, gi, il, and dm. Set $E_1=bk+cf+fh$ and $E_2=gi+il+dm$. The continuum E_1 may be regarded as the union of two continua, bk and $cf+fh+hk$, extending from ae to jn and having a common segment hk of length 1/4. A similar statement holds for E_2. The continua E_1 and E_2 divide the square into five "strips," of which two, $bcfh$ and $ligm$, meet only one side of the square and are called *incomplete*. The other three extend from ae to jn and are called *complete*; two of them are rectangles, $R_1=abkj$ and $R_2=denm$, and the third is a polygon $P_1=$

* Z. Janiszewski, *Sur les continus irréductibles entre deux points*, JOURNAL DE L'ÉCOLE POLYTECHNIQUE, (2), vol. 16 (1926), p. 125.

cdgilkhf; all three have a uniform width of 1/4 measured parallel to the x-axis.

The first step in the second stage is to repeat this construction in the rectangle R_1 with certain modifications. The points corresponding to b, c, d, etc., are: $b'=(1/16, 0)$, $c'=(1/8, 0)$, $d'=(3/16, 0)$, $f'=(1/8, 1/8)$, $g'=(3/16, 1/8)$, $h'=(1/16, 7/8)$, $i'=(1/8, 7/8)$, $k'=(1/16, 1)$, $l'=(1/8, 1)$, $m'=(3/16, 1)$. The essential thing to notice is that the segments $c'f'$, $d'g'$, $h'k'$, and $i'l'$ each have a length equal to one-half the length of the corresponding segments cf, dg, hk, and il. We now have two new continua E_3 and E_4 which have the same properties as E_1 and E_2 except that the common segments $h'k'$ and $d'g'$ have a length of $\frac{1}{2}hk=\frac{1}{8}$. They divide R_1 into five strips, of which three are complete and have a uniform width of 1/16.

If each point (x, y) of E_3 and E_4 is moved to the position (x', y') defined by $x'=x+1/4$ if $y\geqq 3/4$, $x'=x+1/4+\frac{1}{2}(\frac{3}{4}-y)$ if $1/4\leqq y\leqq 3/4$, $x'=x+1/2$ if $y\leqq 1/4$, and $y'=y$, the images E_5 and E_6 of E_3 and E_4 will be two continua dividing P_1 exactly as E_3 and E_4 divide R_1. Likewise the transformation $x'=x+3/4$, $y'=y$ gives two continua E_7 and E_8 dividing R_2. Each of the continua E_5, E_6, E_7, and E_8 is the union of two lines or broken lines extending from *ae* to *jn* and having a common segment of length 1/8. We now have eight continua, $E_1, E_2, \cdots, E_8$, which define nine complete strips, each of width $1/4\times 1/4=1/16$.

Now repeat the construction of the second stage in these nine strips, again making the segments corresponding to $c'f'$, $d'g'$, etc., one-half as long as before. This will give us 18 more continua, $E_9, E_{10}, \cdots, E_{26}$, each of which is the union of two lines or broken lines extending from *ae* to *jn* and having a common segment of length $1/2\times 1/8=1/16$. These divide the nine complete strips of the second stage into 18 incomplete strips and 27 complete strips, each of the latter having a width of $1/4\times 1/16=1/64$.

If this process is carried on indefinitely, the width of the complete strips converges to zero and we get an enumer-

able set of continua E_n of the type described above, whose common segments have a length converging to zero as n increases. If B is the sum of the sets E_n, $\bar{B}$ is the desired continuum C. The improper limiting points of B constitute a non-enumerable set of simple arcs joining ae and jn. Thus C is the sum of an infinity of continua joining ae and jn, which we may call the *elements* of C. We distinguish these by calling the sets E_n elements of the first class, and the simple arcs elements of the second class.

That C is a continuum and is irreducible between any point α of aj and any point β of en is easily deduced from well known theorems. The proof that C cannot be expressed as the union of two continua C_1 and C_2, which have but a single point in common, is briefly as follows. Neither C_1 nor C_2 contains points of both aj and en; let aj be a part of C_1 and let en be a part of C_2. Let x be a point of $C_1 \cdot C_2$ and lie on an element F of C. Then C_1 contains all the elements at the left of F and C_2 contains all those at the right; let C_1^* denote the sum of the first set and C_2^* that of the second. If F is one of the elements E_n, one of the two continua forming E_n lies in $\bar{C}_1^*$ and the other in $\bar{C}_2^*$. But these have a common segment of length greater than zero; hence $C_1 \cdot C_2$ contains a continuum which is not a point. If F is an element of the second class, F lies in both $\bar{C}_1^*$ and $\bar{C}_2^*$, and hence in $C_1 \cdot C_2$. Thus C cannot be expressed as the union of two continua which do not have at least one continuum which is not a point in common.

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ON A FUNDAMENTAL FORMULA IN THE THEORY OF CLASS-NUMBER RELATIONS*

BY E. T. BELL

1. *The Fundamental Formula.* In a series of memoirs *Sur les relations entre les nombres des classes des formes quadratiques et positives* written about seven years ago but only now in course of publication,† Professor J. Ouspensky, perfecting Liouville's unpublished method for class-number relations (Liouville stated only his results), obtains by simple, uniform processes applied to a single formula "not only all the relations hitherto known, with the possible exception of the very general results of Hurwitz, but also many others which appear to be new." The formula in question is (loc. cit., p. 604),

$$\sum[f(d+\delta,\nu,d-\delta)-2f(\delta-2\nu,d+\nu,2d-\delta+2\nu)]$$

$$(1)\qquad =\epsilon(m)\sum_{j=1}^{2m^{1/2}-1}[f(2m^{1/2},\ m^{1/2}-j,\ 2m^{1/2}-2j)-2f(2m^{1/2}-j,\ m^{1/2},\ 2m^{1/2}-j)],$$

where $\sum$ on the left refers to all integers $\nu\gtreqless 0$, $d>0$, $\delta>0$ such that, for m an arbitrary constant integer >0,

$$(2)\qquad m=\nu^2+d\delta\ ;$$

$\epsilon(t)=1$ or 0 according as t is or is not the square of an integer >0, and $f(x, y, z)$ is a single-valued function whenever its arguments are simultaneously integers $\gtreqless 0$, subject to the parity conditions

$$(3)\qquad \begin{cases} f(-x,y,z)=-f(x,y,z), & f(0,y,z)=0,\\ f(x,-y,-z)=f(x,y,z), \end{cases}$$

beyond which f is entirely arbitrary.

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* Presented to the Society, San Francisco Section, June 12, 1926.

† BULLETIN DE L'ACADÉMIE DES SCIENCES DE RUSSIE, 1925, pp. 599–620, 763–784; ibid., 1926, pp. 25–38, 175–196, 327–348.

It is truly remarkable that an identity so simple as (1) should unite a great body of heterogeneous theorems, many of which were first found by abstruse analysis, and reveal their common origin in a matter of elementary arithmetic. In a further *Note sur le nombre des représentations des nombres par une somme d'un nombre de carrés*,* the author announces that the same "formule vraiment fondamentale" yields also the general expression of Boulyguine† for the number of representations of a given integer by any even number of squares, and he gives the extremely simple developments for 2, 4, 6, 8, 10, 12 squares. Further, as remarked by the author, it is evident to anyone conversant with this field that the applications of (1) to class-number formulas and to representations as sums of squares are but a small part of the specific arithmetic facts implicit in the general identity.

Professor Ouspensky's proof of (1) is purely arithmetic and leaves nothing to be desired in the way of elementary simplicity. Believing however that even an elementary proof by strictly arithmetic methods is of a far higher order of difficulty, because such a proof is in general a thing in itself with but slight indications of possible generalizations, than is an algebraic proof employing more advanced analysis, I shall offer in § 2 a proof of (1) of the latter kind.‡ This proof, equally as elementary from the algebraic point of view as is the author's from the arithmetic, refers the theorem to membership in an infinite class of allied results, all of

* Ibid., 1925, pp. 647–662.

† Bulletin de l'Académie des Sciences de St. Pétersbourg, April, 1914.

‡ It is not here a question of rigor, but of practicability, in agreement with the following remarks by H. J. S. Smith (Collected Papers, vol. 1, p. lxxiv) on the formal perfection of Gauss' synthetic proofs: "Every assertion that is made is fully proved; · · · but when we have finished the perusal, we soon begin to feel that our work is but begun · · ·. No vestige appears of the process by which the result itself was obtained, perhaps not even a trace of the considerations which suggested the successive steps of the demonstration."

which are immediate consequences of Jacobi's formula for the multiplication of four elliptic theta functions or, if preferred, of the equation of three terms in elliptic functions, combined with the arithmetic expansions of the doubly periodic functions of the second kind (Hermite's terminology). It is a striking fact that Ouspensky's far-reaching theorem is one of the very simplest of all those concerning arbitrary odd or even functions in *more than two* variables which are implied by Jacobi's theta formula, and which may be elicited from it by the most elementary algebraic considerations.

In a paper not yet published concerning what may be called the inverse method of paraphrases* I have shown that any theorem whatever relating to arbitrary odd or even functions in any number of variables of the same general character as (1), in which the variables are linear functions of the indeterminates representing a set of integers in a system of quadratic forms, is implied by Jacobi's theta formula and the expansions of the doubly periodic functions of the second kind. The doubly periodic functions of the rth kind, $r>1$, also lead to similar results; r determines the type of arithmetic functions occurring as coefficients of the arbitrary functions; for any r the theorems are of essentially one kind, viz., they relate to functions of integers arbitrary except as to parity. I have further shown, in the paper mentioned, how any given result of this kind can be traced immediately by a straightforward, uniform method to its simplest equivalent identity in theta functions. The proof in § 2 was constructed in this way; it will be sufficient here to give the proof directly. Having obtained such an algebraic proof of any theorem we can then extend the theorem indefinitely by multiplying its generating theta identity by functions of theta functions or of circular functions, and reducing, or by numerous other devices of

* For the direct method, used below in §2, see TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), p. 1.

elementary algebra that at once suggest themselves. This is the principal advantage of the inverse method; from the proof by its means of a known theorem the given result can be extended at will. It thus appears that any such result as (1) is included as a special case in each of an infinity of similar theorems relating to functions of 4, 5, 6, $\cdots$ variables. Conversely, starting from any theta identity we can paraphrase it directly, and by specializing the arbitrary functions in the result, deduce an indefinite number of specific arithmetic theorems. Thus the two methods of paraphrase, the inverse and the direct, are complementary.

2. *Algebraic Proof of* (1). In the identity

$$\begin{vmatrix} a_3 & b_3 & c_3 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0, \tag{4}$$

take

$$a_j = \vartheta_j(2x + y + 2z), \quad b_j = \vartheta_j(2x - y - 2z), \quad c_j = \vartheta_j(y),$$

and use

$$\vartheta_1(u)\vartheta_1(v) = \vartheta_3(u+v, q^2)\vartheta_2(u-v, q^2) - \vartheta_2(u+v, q^2)\vartheta_3(u-v, q^2)$$

to reduce the result. Then, dividing throughout by

$$\vartheta_1(x + z,\ q^{1/2})\vartheta_1(x - z,\ q^{1/2})\vartheta_1(y + 2z,\ q^{1/2}),$$

and introducing the doubly periodic function of the second kind indicated by the resulting identity, namely

$$\phi_{111}(u, v)\,\frac{\vartheta_1'\vartheta_1(u+v)}{\vartheta_1(u)\vartheta_1(v)},$$

we find

$$\begin{aligned} &\phi_{111}(x + z,\ x - z,\ q^{1/2})\vartheta_3(y) \\ &= \phi_{111}(x - z,\ y + 2z,\ q^{1/2})\vartheta_3(2z - y - 2z) \\ &\quad + \phi_{111}(x + z,\ -y - 2z,\ q^{1/2})\vartheta_3(2x + y + 2z), \end{aligned} \tag{5}$$

where it is necessary (as in the theta functions) to indicate that the parameter in ϕ_{111} is $q^{1/2}$ not q as it is in $\phi_{111}(u, v)$

and $\vartheta_3(y)$. This is the simplest identity which is formally equivalent to (1), in the sense that each of (1), (5) implies the other. As stated in §1, (4) and the proper theta substitution for a_j, b_j, c_j to yield this formal equivalent of (1) are indicated immediately by the inverse method of paraphrase by simple inspection of (1). In a complete exposition of the theory it would be unnecessary to proceed beyond (5) in order to have the proof of (1). Here, however, we must apply the direct method and verify that (5) actually does imply (1). It remains then to substitute in (5) the series for ϑ_3,

$$\vartheta_3(u) = \sum q^{\nu^2} \cos 2\nu x,$$

where $\sum$ refers to all integers $\nu \gtrless 0$,

$$\phi_{111}(u,v,q^{1/2}) = \operatorname{ctn} u + \operatorname{ctn} v + 4\sum q^n\Big[\sum \sin 2(du+\delta v)\Big]$$

where the outer $\sum$ refers to all integers $n>0$ and the inner to all pairs (d,δ) of conjugate divisors $d, \delta>0$ of n such that $d\delta=n$, in order to reach (1) by paraphrasing the resulting trigonometric identity obtained on equating coefficients of q.

With $\sum$, m, d, δ as in (1), (2), we find at once

$$\begin{cases} 2\sum\big[\sin 2\{(d+\delta)x+(d-\delta)z\}\cos 2\nu y \\ \quad -\sin 2\{dx+\delta y-(d-2\delta)z\}\cos 2\nu(2x-y-2z) \\ \quad -\sin 2\{dx-\delta y+(d-2\delta)z\}\cos 2\nu(2x+y+2z)\big] \\ = \epsilon(m)\big[\operatorname{ctn}(x+z)\{\cos 2m^{1/2}(2x+y+2z)-\cos 2m^{1/2}y\} \\ \quad +\operatorname{ctn}(x-z)\{\cos 2m^{1/2}(2x-y-2z)-\cos 2m^{1/2}y\} \\ \quad +\operatorname{ctn}(y+2z)\{\cos 2m^{1/2}(2x-y-2z) \\ \quad -\cos 2m^{1/2}(2x+y+2z)\}\big]. \end{cases} \tag{6}$$

By directly paraphrasing reduced forms of (6), we can derive immediately *seven* theorems similar to (1), one of which is (1), for functions of the respective parities* $p(1^2|1)$

* For the p(|) notation, see TRANSACTIONS OF THIS SOCIETY, loc. cit., p. 2.

(three theorems), $p(0|1^3)$ (one theorem), $p(2|1)$ (three theorems), all of which are contained as special cases in the result for parity $p(0|1^3)$, which thus appears as the simplest generalization of (1). Our object here being merely the proof of (1), we note that in order to reach (1) all trigonometric products in (6) must be reduced to the type $\sin ax \cos (by+cz)$, since this is the circular function of x, y, z having the same parity with respect to x, y, z as $f(x, y, z)$ in (1). As all details of the necessary reduction are common routine in the method of paraphrase, we need give only an outline of the successive steps.

On the left of (6) trigonometric products are separated into sums and recombined. Thus the first line becomes

$$\begin{aligned} 2\sum \sin 2\{(d+\delta)x + (d-\delta)z\} \cos 2\nu y \\ = \sum [\sin 2\{(d+\delta)x + \nu y + (d-\delta)z\} \\ + \sin 2\{(d+\delta)x - \nu y - (d-\delta)z\}], \end{aligned}$$

since d, δ may be interchanged under $\sum$, and this

$$= 2\sum \sin 2(d+\delta)x \cos 2\{\nu y + (d-\delta)z\},$$

which is in the required form. After similar separation of the second and third lines the sines whose arguments are the respective sums of the given arguments are recombined, and so for the differences. The result is the sum of

$$-2\sum \sin 2(d+2\nu)x \cos 2\{(\delta-\nu)y + (2\delta - d - 2\nu)z\},$$

$$-2\sum \sin 2(d-2\nu)x \cos 2\{(\delta+\nu)y + (2\delta - d + 2\nu)z\},$$

and hence it is twice either, since ν may be replaced by $-\nu$ under $\sum$. To reduce the right of (6) we must first eliminate the cotangents, as always in paraphrasing. For this we combine the differences of cosines into products of sines, obtaining three products of the type ctn $u \sin 2au \sin 2bv$ where a, b are integers >0, after changing the limits of summation for ν $(-\infty$ to $\infty)$ to 0 to ∞, by combining terms

equidistant from $\nu=0$ in the usual way. To these products we apply

$$\operatorname{ctn} u \sin 2au = \sum_{j=0}^{2a-1} \cos 2(a-j)u,$$

separate products $\cos 2(a-j)u \sin bv$ into sums, and proceed as above in reducing the left. Finally, replacing x, y, z by their halves, and noting that the lower limit $j=0$ may be changed to $j=1$ since $j=0$ causes the summand to vanish identically, we find

$$\begin{aligned}\sum[&\sin(d+\delta)x \cos\{\nu y + (d-\delta)z\} \\ &- 2\sin(d+2\nu)x \cos\{(\delta-\nu)y + (2\delta-d-2\nu)z\}] \\ = \epsilon(m) \sum_{j=1}^{2m^{1/2}-1} [&\sin 2m^{1/2}x \cos\{(m^{1/2}-j)y + (2m^{1/2}-2j)z\} \\ &- 2\sin(2m^{1/2}-j)x \cos\{m^{1/2}y + (2m^{1/2}-j)z\}].\end{aligned}$$

Noticing that ν may be replaced by $-\nu$ and that d, δ may be interchanged in the second term on the left we see that the last identity paraphrases directly into (1).

By bringing to bear on (5) all of the powerful machinery of the theta functions, including the theory of transformation, we can obtain from it an inexhaustible supply of theorems similar to (1), some of which, it is likely, it would be difficult to find by strictly arithmetic methods although all, no doubt, could be proved by such means with the exercise of sufficient ingenuity.

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ON THE FUNCTIONAL EQUATION $f(x+y)=f(x)+f(y)$*

BY MARK KORMES

Fréchet,† and later Blumberg‡ and Sierpinski,§ have demonstrated that the solution of the functional equation

$$f(x+y)=f(x)+f(y) \tag{1}$$

which is measurable, has the form $A \cdot x$, where A denotes a constant. In this note the following theorem is proved.

THEOREM I. *Every solution of the functional equation* (1) *which is bounded on a set of positive measure is of the form* $A \cdot x$.

The proof depends on a theorem of Steinhaus¶ which can be stated as follows.

LEMMA. *The set arising by arithmetic summation (addition of abscissas) of a set of positive measure, contains an interval.*||

Since $f(x)$ is bounded on a set of positive measure, $f(x+y)$ is bounded on an interval, and therefore $f(x)$ must be of the form $A \cdot x$ according to a theorem of Darboux.

THEOREM Ia. *The statement of Theorem* I *remains true if* $f(x)$ *is bounded on a set whose interior measure is positive.*

If the interior measure of a set A is $a>0$, then there exists** a measurable sub-set of A whose measure is equal to a (>0).

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* Presented to the Society October 31, 1925.

† M. Fréchet, L'ENSEIGNEMENT MATHÉMATIQUE, vol. 15 (1913), p. 390.

‡ Blumberg, *Convex functions*, TRANSACTIONS OF THIS SOCIETY, vol. 20, p. 41.

§ W. Sierpinski, FUNDAMENTA MATHEMATICAE, vol. 1, p. 116.

¶ H. Steinhaus, FUNDAMENTA MATHEMATICAE, vol. 1, p. 99.

|| The proof of this lemma will be a part of a paper entitled *On arithmetic summation of point sets.*

** C. Carathéodory, *Vorlesungen über Reelle Funktionen*, p. 261.

To this subset Theorem I can be applied and thus Theorem Ia is established.

Theorem I establishes a far more general condition than the one given by Fréchet, Blumberg, and Sierpinski. The following remarks show that this condition is incisive. The condition $m_i(M)^* > 0$ is essential, since there exist non-measurable solutions of (1) which are continuous on a set H, where, for every interval δ,

$$m_i(H \cdot \delta) = 0, \qquad m_e(H \cdot \delta) = \delta.$$

Let B denote a hamelian basis-set of all real numbers. If b is a number of B, we define a solution of the functional equation (1) as follows:

$$f(x) = 0, \quad \text{for the numbers of the set } (B - b);$$
$$f(x) = 1, \quad \text{for } x = b;$$
$$f(x+y) = f(x) + f(y), \quad \text{for all real numbers.}$$

In this way $f(x)$ is completely defined. Let us denote by H the set of all points where $f(x) = 0$. If we denote by H^c the set of all numbers $x+c$, where x assumes all values of H we have then $H \cong H^{ab}$, where the symbol $\cong$ means congruent, and α is a rational number. Then we have

$$(H^{ab} \cdot H^{a'b}) = 0,$$

if $\alpha \neq \alpha'$, and

$$K = \sum_{\alpha} H^{ab},$$

where K denotes the continuum. Therefore we must have,† for every interval δ,

$$m_i(H \cdot \delta) = 0, \qquad m_e(H \cdot \delta) = \delta.$$

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* The symbol $m_i(M)$ shall signify the interior measure of M, $m_e(M)$ the exterior measure of M.

† For suppose $m_i(H \cdot \delta) > 0$. There must exist then a measurable subset $P \subset H$ so that $m(P) > 0$. We would have $m(P^{ab}) = m(P) > 0$. On the other hand it can be shown easily, that then there exists a rational number α_1, so that $m(P^{\alpha_1 b} \cdot P) = a > 0$, but this is impossible, since $(P^{\alpha_1 b} \cdot P) = 0$ because $P \subset H$, $P^{\alpha_1 b} \subset H^{\alpha_1 b}$, and $(H \cdot H^{\alpha_1 b}) = 0$. We must have therefore $m_i(H \cdot \delta) = 0$. See also M. Kormes, *Treatise on basis-sets* (Columbia University dissertation, not yet published), Theorem VIII.

Since the function $f(x)$ is everywhere 0 on the set H, it is bounded and continuous.

There exist non-measurable solutions of (1), which are continuous on a perfect set P, where $m(P)=0$. Let P be the set of all numbers z of the form

$$z=\frac{x_1}{10x_1!}+\frac{x_2}{(10^2x_2+10x_1)!}+\cdots$$

$$+\frac{x_n}{(10^nx_n+10^{n-1}x_{n-1}+\cdots+10x_1)!}+\cdots,$$

where every x_n is either 1 or 2. There cannot exist then any relation of the form

$$\sum_{\lambda} r_\lambda z_\lambda = 0$$

between the numbers z of the set P,* where r_λ denotes a rational number, and in every case only a finite number of r_λ are different from 0. The numbers of P constitute a subset of a basis-set B of all real numbers.† The existence of such basis-set was demonstrated in another paper.‡

We define now a solution of the functional equation (1) in the following way:

$f(x)=0$, for all numbers of P;

$f(x)=1$, for all numbers of $B-P$;

$f(x+y)=f(x)+f(y)$ for all numbers of the continuum K.

But this defines $f(x)$ completely, and it is clear that $f(x)$ is non-measurable and continuous on the perfect set B.

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* M. Kormes, *Treatise on basis-sets.*

† To construct a basis-set B which has a given set P as a subset we proceed in the following way. We well-order the continuum K in such a way that the numbers of P precede all other numbers. The set $(K-P)$ is not empty, and since the set P is not the entire basis-set of K, there must be a first number a_1 of $(K-P)$ which cannot be represented by numbers of P in a linear way. If we consider the set $P_1=P+a_1$ and reason in the same way as above, we obtain a basis-set B of the continuum K. See also M. Kormes, *Treatise on basis-sets.*

‡ M. Kormes, loc. cit.

From Theorem I, the Fréchet-Sierpinski theorem* can be deduced immediately.

THEOREM II. *Every solution of* (1) *which is measurable has the form* $A \cdot x$.

In fact, suppose that $f(x)$ is a solution of (1), and that $f(x)$ is measurable. Then there exists a perfect set P, where $m(P)>0$, and $f(x)$ is continuous on P. Being finite, $f(x)$ must be bounded on P, and Theorem II is a simple consequence of Theorem I.

Theorem I can be generalized for functional equations in n variables. A proof for two variables will be given below and it is quite analogous for $n(>2)$ variables.

THEOREM III. *Every solution of the functional equation*

$$f(x+u,\ y+v) = f(x,y) + f(u,v) \tag{2}$$

where x, y, u, v *denote real numbers, which has the property that* $f(x, 0)$ *is bounded on a measurable set* M_x, *where* $m(M_x)>0$, *and that* $f(0, y)$ *is bounded on a measurable set* M_y, *where* $m(M_y)>0$, *has the form* $A \cdot x+B \cdot y$.†

We have

$$f(x,y) = f(x+0,\ 0+y) = f(x,0) + f(0,y),$$

where $f(x, 0)$ is the solution of the functional equation

$$f(x+u,\ 0) = f(x,0) + f(u,0),$$

and $f(0, y)$ is the solution of the functional equation

$$f(0,\ y+v) = f(0,y) + f(0,v).$$

According to Theorem I, $f(x, 0)$ has the form $A \cdot x$, where $A=f(1, 0)$; and $f(0, y)$ has the form $B \cdot y$, where $B=f(0, 1)$. Therefore $f(x, y)$ has the form $A \cdot x+B \cdot y$.

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* See second, third, and fourth footnotes on p. 689.

† We can assume also that $m_i\ (M_x)$ and $m_i\ (M_y)>0$, and reason in a way similar to that indicated in the proof of Theorem Ia.

The same reasoning holds if $f(x, a)$ and $f(b, y)$ are bounded in M_x and M_y, respectively. We have then $f(x, a) = f(x, 0) + f(0, a)$ and $f(b, y) = f(0, y) + f(b, 0)$. Hence $f(x, 0)$ and $f(0, y)$ must be therefore bounded in M_x and M_y, respectively.

From Theorem III, the following theorem can easily be obtained.

THEOREM IV. *Every solution of the functional equation* (2) *which is bounded on a measurable set* M_{xy}, *whose square measure is* $m^{(2)}(M_{xy}) > 0$, *has the form* $A \cdot x + B \cdot y$.

In order to prove this theorem let us suppose that

$$m^{(2)}(M_{xy}) = a > 0.$$

According to a theorem of Fubini,* there must exist then a straight line $y = b$ parallel to the X-axis, and a straight line $x = a$ parallel to the Y-axis, so that $m(M_{xa}) > 0$ and $m(M_{by}) > 0$. Then $f(x, a)$ would be bounded on the set M_{xa}, where $m(M_{xa}) > 0$; and $f(b, y)$ would be bounded on M_{by}, where $m(M_{by}) > 0$. Therefore $f(x, y)$ must have the form

$$A \cdot x + B \cdot y.$$

From Theorem IV, we may state the following theorem.

THEOREM V†. *Every solution of the functional equation* (2) *which is measurable has the form* $A \cdot x + B \cdot y$.

If $f(x, y)$ is a solution of (2) and it is measurable, then there exists a perfect set P, where $m^{(2)}(P) > 0$, and $f(x, y)$ is measurable on P. Since $f(x, y)$ is finite and P is closed, $f(x, y)$ is bounded on P, and we can apply Theorem IV.

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* Theorem of Fubini-Lebesgue; see de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, vol. 2 (2d ed.), pp. 117–120.

† Theorem of Steinhaus-Sierpinski; see Sierpinski, loc. cit.

ON QUANTIFIERS FOR GENERAL PROPOSITIONS*

BY C. H. LANGFORD

General propositions are commonly constructed in terms of the two applicatives "some" and "every." These applicatives occur singly in propositions of the form $(\exists x) \cdot \phi x$, for at least one value of x, ϕx holds, and $(x) \cdot \phi x$, for every value of x, ϕx holds, and in propositions involving more than one variable constituent when these propositions are of the form $(\exists x, \cdots, n) \cdot f(x, \cdots, n)$ or $(x, \cdots, n) \cdot f(x, \cdots, n)$. Whereas, in propositions of the form

$$(\exists x, \cdots, l)\ (y, \cdots, m)\ (\exists z, \cdots, n) \cdots \cdot f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n; \cdots)$$

or

$$(x, \cdots, l)\ (\exists y, \cdots, m)\ (z, \cdots, n) \cdots \cdot f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n; \cdots),$$

each of the applicatives may have a single or a multiple occurrence.† There are, however, in the traditional treatment of general propositions, four quantitative functions of a property $\phi\hat{x}$, viz., *Every x is such that ϕx*, *Some x is such that ϕx*, *No x is such that ϕx*, and *Not-every x is such that ϕx*. These applicatives occur in the formulation of the syllogism in connection with functions of the form $\phi x \supset \psi x$, and they were, it seems, never carried beyond propositions involving a single applicative. In what follows we shall be concerned chiefly to exhibit the formal properties of general propositions and of general propositional functions when they are expressed directly in terms of the quantifiers *no* and *not-every*.

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* Presented to the Society, May 1, 1926.

† All of the functions which occur in the present discussion are first-order functions; their quantified constituents are all variables which denote individuals. This limitation is, however, merely a matter of convenience. The properties of functions to be exhibited are the same whether the quantified constituents denote individuals or functions.

Any function which involves a single occurrence of an applicative may be said to be singly quantified. Thus $(x) \cdot \phi x$ is singly quantified, as is $(\exists x, \cdots, n) \cdot \phi(x, \cdots, n)$. Any function which involves a multiple occurrence of applicatives, the same or different, may be said to be multiply quantified. Thus $(x) : (y) \cdot \phi(x, y)$ is a multiply quantified function, as is $(x)(\exists y)(z) \cdot \phi(x, y, z)$. The first of these functions is doubly quantified, while the second is triply quantified. Functions involving n occurrences of an applicative, the same or different, will be said to be n-tuply quantified. The terminology introduced here is to be understood to apply also to propositions; any value of an n-tuply quantified function is an n-tuply quantified proposition.

$(x) : (y) \cdot \phi(x, y) : \equiv : (x, y) \cdot \phi(x, y)$ and $(\exists x) : (\exists y) \cdot \phi(x, y) :$

$\equiv : (\exists x, y) \cdot \phi(x, y)$.

$(x, \cdots, m) : (y, \cdots, n) \cdot \phi(x, \cdots, m ; y, \cdots, n) :$

$\equiv : (x, \cdots, m; y, \cdots, n) \cdot \phi(x, \cdots, m ; y, \cdots, n)$

and

$(\exists x, \cdots, m)(\exists y, \cdots, n) \cdot \phi(x, \cdots, m ; y, \cdots, n) :$

$\equiv : (\exists x, \cdots, m; y, \cdots, n) \cdot \phi(x, \cdots, m ; y, \cdots, n)$.

Accordingly, when two occurrences of the same applicative are juxtaposed the variables may be combined under the same applicative. This reduces the degree of quantification of the function. In a function no further reducible no two occurrences of the same applicative will be juxtaposed. Such functions will be said to be in reduced form.

$\sim (x) \cdot \phi x \cdot \equiv \cdot (\exists x) : \sim \phi x$ and $\sim (\exists x) \cdot \phi x \cdot \equiv \cdot (x) \cdot \sim \phi x$,

from which it follows that the sign of negation can be removed from before any quantifier.

$\sim (x, \cdots, l)(\exists y, \cdots, m)(z, \cdots, n)$

$\cdots f(x, \cdots, l ; y, \cdots, m ; z, \cdots, n ; \cdots)$

is equivalent to

$(\exists x, \cdots, l)\ (y, \cdots, m)\ (\exists z, \cdots, n) \cdots$

$\sim f(x, \cdots, l ; y, \cdots, m ; z, \cdots, n ; \cdots)$

and, of course,

$$\sim(\exists x, \cdots, l)\ (y, \cdots, m)\ (\exists z, \cdots, n) \cdots \cdot f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$$

is equivalent to

$$(x, \cdots, l)\ (\exists y, \cdots, m)\ (z, \cdots, n) \cdots \sim f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots).$$

Accordingly, to get the contradictory of a proposition involving a single complex quantifier in "some" and "every" change every universally quantified constituent of the proposition into a particular and every particularly quantified constituent into a universal and take the negative of the function.

Let "No x is such that ϕx" be denoted by $[x] \cdot \phi x$ and "No $x, \cdots, n$ are such that $\phi(x, \cdots, n)$" by $[x, \cdots, n] \cdot \phi(x, \cdots, n)$. Then

$$[x] \cdot \phi x \cdot \equiv \cdot \sim(\exists x) \cdot \phi x \cdot \equiv \cdot (x) \cdot \sim \phi x, \text{ and}$$

$$[x, \cdots, n] \cdot \phi(x, \cdots, n) \cdot \equiv \cdot \sim(\exists x, \cdots, n) \cdot \phi(x, \cdots, n) \cdot \equiv \cdot (x, \cdots, n) \cdot \sim\phi(x, \cdots, n).$$

"Not every x is such that ϕx" may be written $\{x\} \cdot \phi x$, and "Not-every $x, \cdots, n$ are such that $\phi(x, \cdots, n)$," $\{x, \cdots, n\} \cdot \phi(x, \cdots, n)$. Accordingly,

$$\{x\} \cdot \phi x \cdot \equiv \cdot \sim(x) \cdot \phi x \cdot \equiv \cdot (\exists x) \cdot \sim \phi x, \text{ and}$$

$$\{x, \cdots, n\} \cdot \phi(x, \cdots, n) \cdot \equiv \cdot \sim(x, \cdots, n) \cdot \phi(x, \cdots, n) \cdot \equiv \cdot (\exists x, \cdots, n) \cdot \sim \phi(x, \cdots, n).^*$$

Since $(x) \cdot \phi x \cdot \equiv \cdot \sim(\exists x) \cdot \sim\phi x$, any proposition in terms of "some" and "every" can be expressed in terms of "some" and "$\sim$". Thus

$$(x)(\exists y)(z)(\exists w) \cdot f(x, y, z, w) \cdot \equiv \cdot \sim(\exists x)\sim(\exists y)\sim(\exists z) \sim(\exists w) \cdot f(x, y, z, w)$$

* The applicative $\{x\}$ was suggested to me by Dr. H. M. Sheffer, as was the notation used in both cases.

and

$$(\exists x)(y)(\exists z)(w) \cdot f(x, y, z, w) \cdot \equiv \cdot (\exists x) \sim (\exists y) \sim (\exists z) \sim (\exists w) \cdot \sim f(x, y, z, w).$$

And since $[x] \cdot \phi x \cdot \equiv \cdot \sim (\exists x) \cdot \phi x$, we have, as an equivalent of the first of these functions, $[x]\,[y]\,[z]\,[w] \cdot f(x, y, z, w)$, and, as an equivalent of the second,

$$\sim [x]\,[y]\,[z]\,[w] \cdot \sim f(x, y, z, w).$$

Let

$$(x, \cdots, l)(\exists y, \cdots, m) \cdots (\exists z, \cdots, n) \cdot f(x, \cdots, l;\ y, \cdots, m; \cdots; z, \cdots, n)$$

be a function in reduced form. In this function the variables of widest scope are quantified universally, and the degree of quantification of the function is even.

$$\begin{aligned} &(x, \cdots, l)(\exists y, \cdots, m) \cdots (\exists z, \cdots, n) \\ &\quad \cdot f(x, \cdots, l\,;\ y, \cdots,\ m; \cdots\ ; z, \cdots, n) \\ \cdot \equiv \cdot &\sim (\exists x, \cdots, l) \sim (\exists y, \cdots, m) \cdots \sim (\exists z, \cdots, n) \\ &\quad \cdot f(x, \cdots, l\,;\ y, \cdots, m\,; \cdots\ ; z, \cdots, n) \\ \cdot \equiv \cdot &[x, \cdots, l][x, \cdots, m] \cdots [z, \cdots, n] \\ &\quad \cdot f(x, \cdots, l\,; y, \cdots, m\,; \cdots; z, \cdots, n). \end{aligned}$$

This last function is expressed in terms of the applicative [ ] alone, and it does not involve "$\sim$" in the quantifier. Every change of bracket has the force of a change of quantity from universal to particular or from particular to universal. This is so because the force of [ ] is negative. For example, the proposition "Every element has a successor," which may be written $(x) : (\exists y) \cdot x < y$, is equivalent to "No element is without a successor," which may be written $[x] : [y] \cdot x < y$, there is *no* x such that *no* y is such that $x < y$. Propositions whose variables of widest scope are quantified universally do not entail existence; that is to say, they would be true if there were not at least one element within the range of significance of the variables.

Let $(x, \cdots, l)\,(\exists y, \cdots, m) \cdots (z, \cdots n) \cdot f(x, \cdots, l;\ y, \cdots, m; \cdots; z, \cdots, n)$

be a function in reduced form. Its degree of quantification is odd and its variables of widest scope are quantified universally.

$$\begin{aligned}&(x, \cdots, l)(\exists y, \cdots, m) \cdots (z, \cdots, n)\\ &\cdot f(x, \cdots, l\,; y, \cdots, m\,; \cdots\,; z, \cdots, n)\\ \cdot \equiv \cdot &\sim (\exists x, \cdots, l) \sim (\exists y, \cdots, m) \cdots \sim (\exists z, \cdots, n)\\ &\cdot \sim f(x, \cdots, l\,; y, \cdots, m\,; \cdots\,; z, \cdots, n)\\ \cdot \equiv \cdot &[x, \cdots, l][y, \cdots, m] \cdots [z, \cdots, n]\\ &\cdot \sim f(x, \cdots, l; y, \cdots, m\,; \cdots\,; z, \cdots, n).\end{aligned}$$

Here "$\sim$" appears before the elementary function, but it is not involved in the quantifier. For example, in connection with serial relations, the proposition "Every element has an immediate successor" may be expressed by

$$(x)(\exists y)(z) : x < y : x < z \cdot y < z \cdot \ \vee \cdot z < x \cdot z < y,$$

which is equivalent to

$$[x][y][z] : \sim (x < y : x < z \cdot y < z \cdot \ \vee \cdot z < x \cdot z < y).$$

Let $(\exists x, \cdots, l)\,(y, \cdots, m) \cdots (z, \cdots, n)$
$\cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$
express a function in reduced form. The degree of quantification is even and the variables of widest scope are quantified particularly. The function is equivalent to

$$\begin{aligned}&\sim \sim (\exists x, \cdots, l) \sim (\exists y, \cdots, m) \cdots \sim (\exists z, \cdots, n)\\ &\cdot \sim f(x, \cdots, l\,; y, \cdots, m\,; \cdots\,; z, \cdots, n),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\sim [x, \cdots, l][y, \cdots, m] \cdots [z, \cdots, n]\\ &\cdot \sim f(x, \cdots, l; y, \cdots, m\,; \cdots\,; z, \cdots, n).\end{aligned}$$

Here "$\sim$" appears before the entire function. This is necessary since [ ] is universal and the function to be expressed is particular in respect of its variables of widest scope. Propositions of this form entail existence; they

would be false if there were not at least one element within the range of significance of the variables.

$$(\exists x, \cdots, l)(y, \cdots, m) \cdots (\exists z, \cdots, n)$$
$$\cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \sim\sim(\exists x, \cdots, l) \sim(\exists y, \cdots, m) \cdots \sim(\exists z, \cdots, n)$$
$$\cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \sim [x, \cdots, l][y, \cdots, m] \cdots [z, \cdots, n)$$
$$\cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n).$$

It is clear that if a function is to be expressed in terms of [ ] it must be in reduced form. Functions expressed in terms of "some" and "every" have a multiplicity of forms which do not appear in functions on [ ]. Thus

$$(x)(y)(\exists z)(\exists w) \cdot f(x,y,z,w) \cdot \equiv \cdot (x,y)(\exists z)(\exists w)$$
$$\cdot f(x,y,z,w)$$
$$\cdot \equiv \cdot (x)(y)(\exists z,w) \cdot f(x,y,z,w) \cdot \equiv \cdot (x,y)(\exists z,w)$$
$$\cdot f(x,y,z,w).$$

Any one of these functions is equivalent to

$$[x,y][z,w] \cdot f(x,y,z,w),$$

and there is no other equivalent form in this quantifier; a change of bracket is always significant, and no two functions differing as to degree of quantification can be strictly equivalent.

Since $(\exists x) \cdot \phi x \cdot \equiv \cdot \sim (x) \cdot \sim \phi x$, any function in "some" and "every" can be expressed as a function in "every" and "$\sim$". Thus

$$(x, \cdots, l)(\exists y, \cdots, m) \cdots (\exists z, \cdots, n)$$
$$\cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \sim\sim(x, \cdots, l) \sim(y, \cdots, m) \cdots \sim(z, \cdots, n)$$
$$\cdot \sim f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n),$$

and since

$$\{x, \cdots, n\} \cdot \phi(x, \cdots, n) \cdot \equiv \cdot \sim(x, \cdots, n) \cdot \phi(x, \cdots, n),$$

this function is equivalent to

$$\sim\{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\}$$
$$\cdot \sim f(x, \cdots, l\,;\, y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n).$$

Here we cannot dispense with "$\sim$" as affecting the entire function, since the force of the applicative $\{\ \}$, when it occurs in the first place, is particular, whereas the applicative in the first place of the function to be expressed is universal.

$$(x, \cdots, l)(\exists y, \cdots, m) \cdots (z, \cdots, n)$$
$$\cdot f(x, \cdots, l\,;\, y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n)$$
$$\cdot \equiv \cdot \sim \sim (x, \cdots, l) \sim (y, \cdots, m) \cdots \sim (z, \cdots, n)$$
$$\cdot f(x, \cdots, l\,;y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n)$$
$$\cdot \equiv \cdot \sim \{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\}$$
$$\cdot f(x, \cdots, l\,;y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n).$$

Also,

$$(\exists x, \cdots, l)(y, \cdots, m) \cdots (\exists z, \cdots, n)$$
$$\cdot f(x, \cdots, l\,;\, y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n)$$
$$\cdot \equiv \cdot \sim (x, \cdots, l) \sim (y, \cdots, m) \cdots \sim (z, \cdots, n)$$
$$\cdot \sim f(x, \cdots, l\,;y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n)$$
$$\cdot \equiv \cdot \{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\}$$
$$\cdot \sim f(x, \cdots, l\,;y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n).$$

And

$$(\exists x, \cdots, l)(y, \cdots, m) \cdots (z, \cdots, n)$$
$$\cdot f(x, \cdots, l\,;y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n)$$
$$\cdot \equiv \cdot \sim (x, \cdots, l) \sim (y, \cdots, m) \cdots \sim (z, \cdots, n)$$
$$\cdot f(x, \cdots, l\,;y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n)$$
$$\cdot \equiv \cdot \{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\}$$
$$\cdot f(x, \cdots, l\,;y, \cdots, m\,;\, \cdots\,;\, z, \cdots, n).$$

Heretofore we have been concerned with functions which involve a single complex quantifier, and in which, ac-

cordingly, any two variables have interdependent scopes, and each variable has within its scope the whole of the elementary function. There are two kinds of functions which do not have this form. (1) A function may be such that, although any two variables have interdependent scopes, at least one variable does not have the entire elementary function within its scope. The function

$$(y) : \phi y \cdot \vee \cdot (\exists x) \cdot \psi(x, y)$$

is such a function; or the function

$$(y) : \phi y \cdot \vee \cdot (\exists x) \cdot \psi x.$$

It is to be noted, with regard to this latter function, that since ψx is within the scope of (y), the variable y may occur in ψx. (2) A function may be such that at least two variables have independent scopes. Thus

$$(y) : (\exists x) \cdot \phi(x, y) \cdot \vee \cdot (z) \cdot \psi(y, z)$$

and

$$(y) \cdot \phi y \cdot \vee \cdot (\exists z) \cdot \psi z$$

are such functions.

It is clear that any function can be so expressed that "$\sim$", when it occurs, does not have any quantifier within its scope. For $\sim(x) \cdot \phi x \cdot \equiv \cdot (\exists x) \cdot \sim \phi x$ and $\sim(\exists x) \cdot \phi x \cdot \equiv \cdot (x) \cdot \sim \phi x$, and $\sim(\phi x \cdot \psi y) \cdot \equiv \cdot \sim \phi x \vee \sim \psi y$ and $\sim(\phi x \vee \psi y) \cdot \equiv \cdot \sim \phi x \cdot \sim \psi y$. Now it can be shown that for any function there is a function in a single complex quantifier which is equivalent to it,* so that every function can be *expressed* in this form. We may consider functions of the first kind first. Let ϕx and ψy be functions such that ϕx does not involve the variable y. Then $(y) \cdot \phi x \vee \psi y$ entails $\phi x \cdot \vee \cdot (y) \cdot \psi y$. Conversely, if $\phi x \cdot \vee \cdot (y) \cdot \psi y$, then $(y) \cdot \phi x \vee \psi y$; so that $(y) \cdot \phi x \vee \psi y : \equiv : \phi x \cdot \vee \cdot (y) \cdot \psi y$. Similarly, $(\exists y) \cdot \phi x \vee \psi y : \equiv : \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$. Also $(y) \cdot \phi x \cdot \psi y : \equiv : \phi x : (y) \cdot \psi y$ and $(\exists y) \cdot \phi x \cdot \psi y : \equiv : \phi x : (\exists y) \cdot \psi y$. By a repeated application of these equivalences it is possible to transform any function of the kind (1) into a function which involves a single complex quantifier.

*See *Principia Mathematica*, vol. I, *9.

In dealing with functions of the kind (2), we may require the equivalences just given, but some one or more of the following relations are always necessary: $(x) \cdot \phi x : (y) \cdot \psi y : \equiv : (x) : (y) \cdot \phi x \cdot \psi y$; $(\exists x) \cdot \phi x : (\exists y) \cdot \psi y : \equiv : (\exists x) : (\exists y) \cdot \phi x \cdot \psi y$; $(x) \cdot \phi x \cdot \vee \cdot (y) \cdot \psi y : \equiv : (x) : (y) \cdot \phi x \vee \psi y$; $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y : \equiv : (\exists x) : (\exists y) \cdot \phi x \vee \psi y$. In each of these equivalences, as in those which follow, it must, of course, be true that ϕx does not involve the variable y and that ψy does not involve the variable x. $(x) \cdot \phi x : (\exists y) \cdot \psi y$ entails $(\exists x) : (y) \cdot \phi x \cdot \psi y$ and $(\exists x) : (y) \cdot \phi x \cdot \psi y$ entails $(x) \cdot \phi x : (\exists y) \cdot \psi y$, so that $(x) \cdot \phi x : (\exists y) \cdot \psi y : \equiv : (\exists y) : (x) \cdot \phi x \cdot \psi y$. But it is to be noted that although $(x) \cdot \phi x : (\exists y) \cdot \psi y$ entails $(x) : (\exists y) \cdot \phi x \cdot \psi y$, it is not the case that $(x) \cdot \phi x : (\exists y) \cdot \psi y$ follows from $(x) : (\exists y) \cdot \phi x \cdot \psi y$. This relation will be dealt with in detail presently since it involves a point about which mistakes have frequently been made. $(x) \cdot \phi x) \cdot \vee \cdot (\exists y) \cdot \psi y$ entails $(x) : (\exists y) \cdot \phi x \vee \psi y$ and $(x) : (\exists y) \cdot \phi x \vee \psi y$ entails $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$, so that these functions are equivalent. But, although $(\exists y) : (x) \cdot \phi x \vee \psi y$ entails $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$, $(\exists y) : (x) \cdot \phi x \vee \psi y$ does not follow from $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$. By a repeated application of the foregoing equivalences any function of the kind (2) can be expressed in a form which involves a single complex quantifier.

We have noted that $(\exists y) : (x) \cdot \phi x \vee \psi y$ does not follow from $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$ and that $(x) \cdot \phi x : (\exists y) \cdot \psi y$ does not follow from $(x) : (\exists y) \cdot \phi x \cdot \psi y$. It has been pointed out that propositions of the form $(x) \cdot \phi x$ would be true if there were not at least one individual within the range of significance of the variables. For $(x) \cdot \phi x$ entails $\sim(\exists x) \cdot \sim \phi x$ and $\sim(\exists x) \cdot \sim \phi x$ entails $(x) \cdot \phi x$; $(\exists x) \cdot \sim \phi x$ entails $\sim(x) \cdot \phi x$ and $\sim(x) \cdot \phi x$ entails $(\exists x) \cdot \sim \phi x$. Accordingly, $(x) \cdot \phi x$ and $(\exists x) \cdot \sim \phi x$ are proper contradictories, — $(x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ is necessary and $(x) \cdot \phi x : (\exists x) \cdot \sim \phi x$ is impossible. Now the proposition $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ is not a necessary proposition.

It is empirically significant in that it entails the existence of at least one individual, and this is not a fact which is certifiable on formal grounds alone. If it is not the case that $(x) \cdot \phi x$ would be true if there were not at least one value for x, then $(x) \cdot \phi x$ entails $(\exists x) \cdot \phi x$. Accordingly, $(x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ entails $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$, and this latter proposition is necessary since it follows from a necessary proposition. But it is not the case that $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ is a necessary proposition.

It follows that the function $(x) : (\exists y) \cdot \phi x \vee \psi y$ does not make an existence demand. Any value of this function may be read: It is false that there is at least one value of $(\exists y) \cdot \phi x \vee \psi y$, say $(\exists y) \cdot \phi x_1 \vee \psi y$, such that $(\exists y) \cdot \phi x_1 \vee \psi y$ is false. This will be true if there are no values. On the other hand, any value of the function $(\exists y) : (x) \cdot \phi x \vee \psi y$ may be read: There is at least one value of $(x) \cdot \phi x \vee \psi y$, say $(x) \cdot \phi x \vee \psi y_1$, such that $(x) \cdot \phi x \vee \psi y_1$ is true. This requires at least one value for y. Accordingly, these two functions are not strictly equivalent.

We have shown that any function can be so expressed as to have a single complex quantifier. It follows that every function has either the form

$$(x, \cdots, l)(\exists y, \cdots, m)(z, \cdots, n) \cdots f(x, \cdots, l ; y, \cdots, m ; z, \cdots, n ; \cdots)$$

or the form

$$(\exists x, \cdots, l)(y, \cdots, m)(\exists z, \cdots, n) \cdots f(x, \cdots, l ; y, \cdots, m ; z, \cdots, n ; \cdots).$$

The first of these functions can be expressed in terms of $[\ ]$ alone and the second can be expressed in terms of $\{\ \}$ alone. Consequently, any function can be so expressed as to have one of the forms

$$[x, \cdots, l][y, \cdots, m][z, \cdots, n] \cdots f(x, \cdots, l ; y, \cdots, m ; z, \cdots, n ; \cdots)$$

or

$$\{x, \cdots, l\}\{y, \cdots, m\}\{z, \cdots, n\} \cdots f(x, \cdots, l ; y, \cdots, m ; z, \cdots, n ; \cdots).$$

In accordance with these forms, any function of whatever degree of quantification can be given in terms of a single applicative. One of every pair of mutually contradictory propositions can be given in terms of [ ] alone and the other can be given in terms of { } alone. $[x, \cdots, n] \cdot f(x, \cdots, n)$ is equivalent to $\sim\{x, \cdots, n\} \cdot \sim f(x, \cdots, n)$ and $\{x, \cdots, n\} \cdot f(x, \cdots, n)$ is equivalent to $\sim [x, \cdots, n] \cdot \sim f(x, \cdots, n)$. Similarly, $[x, \cdots, l]$ $[y, \cdots, m]$ $[z, \cdots, n] \cdots f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$ is equivalent to $\sim \{x, \cdots, l\}$ $\{y, \cdots, m\}$ $\{z, \cdots, n\} \cdots \sim f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$ and $\{x, \cdots, l\}$ $\{y, \cdots, m\}$ $\{z, \cdots, n\}$ $\cdots f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$ is equivalent to $\sim[x, \cdots, l]$ $[y, \cdots, m]$ $[z, \cdots, n] \cdots \sim f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$.

The function $(x) \cdot \phi x$ is a generalization of the conjunctive function $\phi x_1 \cdot \phi x_2 \cdot \phi x_3 \cdots \phi x_n$, which involves a finite number of the values of ϕx. Similarly, $(\exists x) \cdot \phi x$ is a generalization of the finite disjunctive function

$$\phi x_1 \vee \phi x_2 \vee \phi x_3 \vee \cdots \vee \phi x_n.$$

In precisely the same way $[x] \cdot \phi x$ is a generalization of

$$\sim\phi x_1 \cdot \sim\phi x_2 \cdot \sim\phi x_3 \cdots \sim\phi x_n.$$

Now this latter function has the same force as Sheffer's "stroke" function $p|q$,* when $p|q$ is rendered $\sim p \cdot \sim q$. Accordingly, $[x] \cdot \phi x$ is a generalization of $\phi x_1|\phi x_2$, when interpreted ϕx_1 is false and ϕx_2 is false. Similarly, $\{x\} \cdot \phi x$ is a generalization of

$$\sim\phi x_1 \vee \sim\phi x_2 \vee \sim\phi x_3 \vee \cdots \vee \sim\phi x_n.$$

This function has the same force as the "stroke" function $p|q$ in its alternative interpretation $\sim p \vee \sim q$; so that $\{x\} \cdot \phi x$ generalizes $\phi x_1|\phi x_2$ in this interpretation.

HARVARD UNIVERSITY

* See H. M. Sheffer, *A set of five independent postulates for boolean algebras*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 481–488.

BIRKHOFF ON RELATIVITY

The Origin, Nature, and Influence of Relativity. By G. D. Birkhoff. New York, Macmillan, 1925. ix+185 pp.

This elementary introduction to the principles underlying the theory of relativity is based on six lectures delivered at the Lowell Institute in the fall of 1923 and on eight lectures delivered nearly a year later at the Southern Branch of the University of California. In the preface the author says "The degree of general interest manifested, even in the two more technical lectures given at Los Angeles, seemed to indicate the desirability of revising, extending, and unifying this material in book form. My friend Professor Hedrick encouraged me to carry out such a project, and the book now lies before the reader as part of the Macmillan series of mathematical publications of which he is editor."

The book is characterized by its insistence upon the current trend of thought towards a deeper unification of science, by the careful analysis which is made of the way in which one abstraction in science grows out of and includes earlier abstractions, by the clarity and non-technical character of the exposition of the underlying principles of relativity, and by the "formulation of a definite theory of the structure of space and time, of matter and electricity."

The great abstractions which lie back of the theory of relativity and are presupposed in its development are expounded briefly and clearly and in a masterly way. A certain important type of relativity involved in the geometry of Euclid is set forth with precision, with a backward reference to the insistence of Pythagoras upon the fundamental importance of number in the study of natural phenomena. One passes naturally from Euclid to the type of relativity which is embodied in the Newtonian mechanics as it has actually been developed. An analysis is made of the character of the space in which Newton's first law of motion can be valid. Then comes Faraday as the foremost among those who have investigated the facts concerning electricity and magnetism. Since he was lacking in mathematical training "it required his mathematical interpreter Maxwell to give Faraday's concretely expressed 'lines of force' an exact and adequate statement." In the formulation by Maxwell of the laws of electricity and magnetism there is involved a more comprehensive doctrine of relativity of natural law than that of Euclid or that of Newton. From the relativistic elements present in the great doctrines of Euclid, Newton, and Faraday-Maxwell, the author proceeds by steps having the character of inevitableness to the special and then to the general theory of relativity. This process of step-by-step development is so set forth that one can not fail to have a vivid impression of the increasing unity of scientific thought and of the way in which one abstraction is built upon and includes those which precede it. It is not a question of discarding the old for the new but a process of developing and extending the old so that it includes the new. In this is

involved implicitly a profound principle of the philosophy of scientific development. These results are achieved in the first chapter and the facts attain an increasing clarity as the exposition proceeds.

The nature of space and time is treated in the second chapter. The third contains an exposition of the old and the new theories of gravitation. These are followed in order by two chapters, one on the experimental tests of relativity and one on some relativistic paradoxes and their explanations. A lucid account and explanation of the difficulties connected with the notion of simultaneity of events is one of the most pleasing features of this analysis of the paradoxes.

These first five chapters are entirely non-technical in their exposition and they put the main ideas of relativity, correctly stated, in a form "not beyond the comprehension of any one possessing a modicum of scientific training."

The essential simplicity of the principles underlying the general theory of relativity is effectively insisted upon. It is not denied that the detailed development of the theory involves complexity. But in this respect the theory of relativity is not alone. This is illustrated by the fact that the determination of the exact mode of vibration of a tuning-fork has never been made. "It is not even possible to explain its general behavior save by elaborate theoretical considerations, nor to specify the fundamental note in advance by any formula."

The sixth and seventh chapters contain the most specific contribution made in the book. In them a definite theory of space and time, of matter and electricity, is formulated. It is practically entirely in these two chapters that mathematical formulàs are to be found; but even in these a significant portion of the contents can be understood by one to whom these formulas are meaningless. The author expects to elaborate this theory more fully in a later publication.

The final Chapter VIII is devoted to an analysis of the philosophical influence of relativity. Much which might be profitably said upon this subject is omitted; but several important conclusions are drawn which one is not accustomed to find elsewhere. In view of the growth of abstractions and their inclusion of previous abstractions, illustrated in the most effective way in the development of the theory of relativity, the author concludes: "Relativity does not suggest then that ideals are relative and shifting, but rather that they will enlarge from time to time." He then adds the following:

"The great variety of ways in which one and the same abstraction may be approached indicates that the real differences of underlying opinion at the social level can only be inferred when there exist practical decisions that are different. Undoubtedly this fact should make for sympathy and tolerance. · · · The true role of the systematized abstraction has become more apparent, and freedom has been won to use a formula or a theory and yet not be enslaved to it."

The author is justified in his hope (p. vi) "that the book will prove useful in a mathematical, physical, or philosophical course covering this field, as well as of interest to the general scientific reader." This should be apparent from what has been said in this review. Moreover, I agree heartily

with the concluding sentence of the preface: "The field is indeed a most fascinating one, and I believe a course of this character must be given in all our colleges and universities if we are to be fair to our students." The reviewer believes that departments of mathematics, where they have competent instructors, should take the lead in organizing such courses. This book could well furnish a part of the basis for such a course.

R. D. CARMICHAEL

DICKSON ON MODERN ALGEBRA

Modern Algebraic Theories. By L. E. Dickson. Chicago, New York, Boston, Sanborn, 1926. ix+276 pp.

At last English speaking students have a clear, concise guide to the essentials of each of the great branches of modern algebra, written by an authority who has himself enriched several of them by his own outstanding contributions. This publication of Professor Dickson's matured synopsis of modern algebraic theories is a notable event for American mathematics in at least two respects. First, the book will doubtless be for many years the vade mecum of successive generations of graduate students seeking to penetrate the wide and increasingly significant domain of modern algebra. Second, it may suggest to other publishers that it pays to serve mathematics by the publication of something better than inflated advertising matter and mediocre sophomore texts which live their two or three semesters and become liabilities. As it is but seldom that an American publishing house shows sufficient interest in the advancement of science to bring out a work of the calibre of this book, it is to be hoped and expected that the mathematical public will express its appreciation of the publisher's farsighted enterprise in a tangible manner.

Linear algebras, and certain other advanced parts of modern algebra are not discussed in the present treatise; for these the reader is referred to the author's *Algebras and their Arithmetics* (soon to appear in amplified form in a German translation). Presupposing only a knowledge of the rudiments of the calculus and of the elementary theory of equations, *Modern Algebraic Theories* gives rapid, crystal-clear introductions to algebraic invariants, "higher algebra" as usually understood in America, the Galois theory of algebraic equations, the theory of finite linear groups including Klein's theory of the icosahedron and equations of the fifth degree, and group characters.

In the first chapter of only 23 pages the leading concepts of algebraic invariants in the non-symbolic notation are laid down so swiftly that on pp. 17–20 the author obtains the fundamental systems of covariants for the binary p-ic, $p<5$, exhibits the syzygies between them, and on page 23 reaches the solution of the quartic equation from the canonical form of the binary quartic, having defined and proved the covariance of Hessians, Jacobians and discriminants on the way, all with a minimum of computation. Only the essentials are treated; the swarms of insignificant minutiae in which less experienced writers revel, are ignored. The second chapter drives

straight ahead, undeflected from the main goal by trivial temptations, until at p. 35 the existence of a fundamental system of covariants of every set of binary forms is proved and on the same page the finiteness of syzygies is established. It is no small achievement to have presented simply and intelligibly so much good mathematics in so little space.

Under the section on "higher algebra" (Chapters III–VI) we find a detailed exposition of linear forms, linear transformations, matrices, quadratic and Hermitian forms, symmetric and Hermitian bilinear forms, invariant factors and elementary divisors, pairs of bilinear, quadratic, and Hermitian forms, all in about 95 pages. The treatment is fresh and in several important respects original with the author. Two novel features may be particularly noted. In Chapter IV the study of two symmetric or two Hermitian bilinear forms is made to depend upon that of two quadratic or two Hermitian forms and the last two problems are treated together by a new, simple device, first introduced in the author's Chicago lectures, so that the theory of the four species of forms, quadratic, Hermitian, symmetric and Hermitian bilinear is unified and presented in but little more space than has hitherto been required for even a concise exposition of any one of the four. Second, in Chapter VI, on pairs of bilinear, quadratic and Hermitian forms, the author presents a new and simpler theory of the equivalence of pairs of such forms. Incidentally exact care is given to all questions of rationality.

This part of higher algebra, long disregarded by mathematical physicists as but another bizarre pastime of the pure mathematicians, having recently become prominent in theories of quanta, has acquired a new significance, and it seems likely that the well trained theoretical physicist may henceforth find it suggestive to accord to algebra a share of the attention which he has lavished on analysis. Matrices, bilinear and Hermitian forms, the highly finished weapons of the pure mathematician, have passed suddenly into the armory of the aggressive questioner of nature. So it has been in the past, times out of number, with abstract theories and so, believe those who give their lives to pure mathematics, will it be in the future.

Naturally the exposition of so extensive a field as that of "higher algebra" in such narrow compass implies close writing and closer reasoning. As everywhere in the book not a word is wasted. On the other hand clarity is not sacrificed to brevity. It is easy to achieve condensation by the Laplacian device of substituting for proof the discouraging falsehood "it is easy to see," but Professor Dickson has not descended to this stale trick of the indolent. No link in the strong chains of reasoning is slurred; the final product is fool-proof. This last does not imply of course that the book is designed for the feeble-minded or the lazy; the best students will find on nearly every page a sentence or two worthy of their sinews. Some of the theories treated—especially in the last half of the book—are inherently difficult for beginners unaccustomed to abstract reasoning; mere expansion will not render the essential nub less hard but will only obscure it in a haze of irrelevancies. Brevity often is the soul of clearness.

The theme in Chapters VII–X is algebraic equations with the Galois theory in the place of honor. Readers approaching this somewhat abstract

theory for the first time will find here a remarkably concrete presentation, leading by easy gradations from the familiar cubic and quartic equations through the first principles of substitution groups to Galois resolvents and the group of an equation, with a short intermediary chapter (VIII) on fields, adjunction and reducibility, reaching finally, after only 40 pages from the elementary beginning of the whole story, the group of the general equation of the nth degree. Chapter X, of 25 pages, disposes of the problem of solvability by radicals, and does so in a fashion which should convince anyone capable of human reason. This chapter and the next may be particularly recommended to those (their tribe is far from extinct) who still receive patiently alleged solutions of the general equation of the fifth degree, duplications of the cube and trisections. The just criticisms of Wantzel's revision of Abel's proof that the general equation of degree >4 is not solvable by radicals, and of Abel's own proof, are not evaded here as in certain other works on algebraic equations, but are fairly stated and, what is more important, answered by sound proofs.

The matter in Chapters VII–X is self-contained, and it is probably the shortest, as it is easily one of the clearest, expositions of the modern theory of algebraic equations. These chapters could be read independently of the rest of the book. The converse is not true, however; certain subsequent sections presuppose for their understanding a thorough mastery of these.

The course in Galois theory here presented is reared from first principles with everywhere a minimum of computations, and it is well within the range of able undergraduates. Unlike some phases of modern algebra this theory demands for its comprehension a native capacity for abstract thought rather than technical facility in ingenious manipulations. It is thus one of the infallible touchstones for separating the born mathematicians from the goats. There is no good reason why the ambitious stripling should not test his powers early on the Galois theory instead of deferring acquaintanceship with mathematics as mathematicians like it till the first year of graduate work. To such Professor Dickson's short, well-knit treatise on this great theory may be unreservedly recommended as the adequate means for self-instruction.

In the six pages of Chapter XI the student is discouraged from attempts to trisect all angles, and he is shown exactly what regular polygons can be constructed by the methods of euclidean geometry. Here a judicious use of the previously developed group theory greatly shortens the proofs.

Chapter XII, concerning Tschirnhaus transformations, the Bring-Jerrard normal form, and Brioschi's normal form of quintic equations, is preliminary to XIII, the first part of which presents with remarkable simplicity the essentials of Klein's *Ikosaeder*. Material simplifications due to the author make the treatment particularly fresh and pleasing.

Presupposing a minimum of previous knowledge the author gives (Chapter XIII) in 29 pages a direct derivation of each finite group of linear transformations on two variables, thus acquainting the reader with the outstanding landmarks of Klein's masterpiece so far as they relate to algebra. Here we see algebra at its best, bold of imagination, untrammelled by mean calculations and mighty in the mastery of age-old problems. That

so familiar a thing as the icosahedron which Plato doted on almost to the verge of superstition should hold the true key to the riddle of quintic equations, sought for in vain by the keenest algebraists of the eighteenth and early nineteenth centuries, and grasped first in its entirety only by the many-minded Klein after Hermite and Kronecker had done great things with an older magic, must ever remain one of the perfect revelations of pure science. This fascinating chapter XIII includes full discussions of the related form problems and the Galois group of the icosahedral equation; the solution of the general quintic is carried up to the point prescribed by the limits of the book, namely to that of the icosahedral equation, where algebra merges into the theories of special functions. It will be a cold reader indeed who is not inspired by this presentation to proceed the remaining steps of the way and see how the quintic equation is finally solved explicitly in a domain beyond algebra.

Early in Chapter XIV, a sequel to the preceding, the representation of any substitution group as a linear group is established. Inter alia this chapter discusses group matrices, group characters, the computation of the latter and their application to the former. Probably the strongest attraction which this chapter will have for most readers is the easy introduction which it offers to the powerful method of group characters, following (with simplifications) the exposition of Schur, as this efficient tool is none too accessible for beginners in the classic memoirs. As pointed out by the author this theory has unearthed treasures for finite groups not previously known, so that it is not merely an elegant retelling of an old story. It is perhaps as close to an algorithm as any method at present possessed by discrete groups, and it deserves to be mastered by a greater number of students than now use it.

Throughout the book numerous short sets of approachable exercises are provided for the reader to test his mastery of the several theories. These have been judiciously selected and are not mere pockets of hard nuts from crabbed papers for which there was no room in the text. There are also occasional historical notes where the matter is novel or of sufficient interest to justify an allusion, and five extremely helpful lists of references for further reading, much of it in English.

Professor Dickson has given us a gem of mathematical exposition, glowing with the fire of the master, and touched with the creative spark of the living mathematician. This book could not have been written by one not himself an original algebraist. The reader who will diligently understand what he reads will see that algebra is not the wretched hodge podge of shifty tricks which some writers would make it, but a compact science and a high art.

E. T. BELL

WHITEHEAD AND RUSSELL'S PRINCIPIA MATHEMATICA

Principia Mathematica. By Alfred North Whitehead and Bertrand Russell, Volume I. Second Edition. Cambridge, University Press, 1925. xlvi+674 pp.

The second edition of Volume I of Whitehead and Russell's *Principia Mathematica* leaves the first edition intact (except for minor changes), but adds some new sections. The new sections—an introduction and three appendices—are chiefly devoted to a restatement of the logical theories of the authors in the light of the reduction in the number of primitives in the *Principia* by Sheffer and Nicod and in the light of the views of Wittgenstein (expressed in his remarkable *Tractatus Logico-Philosophicus*) on propositions and propositional functions.

The authors announce in the new Introduction the main improvements which they find necessary to make in their logic. These improvements I may list as follows: the dropping of the distinction between "real" and "apparent" variables; the dropping of the primitive idea "assertion of a propositional function"; the reading, on all occasions, of "$\vdash \cdot fp$" as "$\vdash \cdot (p) \cdot fp$"; the dropping of the primitive proposition *1·11. The authors, also, apparently abandon their "axiom of reducibility," because "clearly it is not the sort of axiom with which we can rest content," though, because of this abandonment, "there is, so far as we can discover, no way by which our present primitive propositions can be made adequate to Dedekindian and well-ordered relations." I have not included in the above list of changes the Sheffer-Nicod simplification of the primitives of the *Principia*. This simplification, elegant as it is from a certain point of view, is not a logical necessity. The general logical make-up of the old *Principia* is not affected by the revised edition. The authors begin with a set of primitives in the logic of "elementary" propositions, whose independence (and consistency) are left unproved because "the ordinary methods of proving independence are not applicable, without reserve, to fundamentals," and from these primitives, together with primitives introduced later, they aim to derive, in a definitely restricted way, the rest of logic and all mathematics.

When one considers the caliber of our authors and the fact that the *Principia* has occupied a prominent place on mathematical shelves for fourteen years, one wonders that the book has influenced mathematics so little. Of course, a partial explanation lies in the magnitude and structure of the *Principia*. Volume I, which is only one of four royal octavo volumes, which deals only with mathematical logic and matters introductory to the theory of cardinals, contains over 600 pages (over 700 in the new edition), has over 400 different symbols (some of them representing extremely subtile ideas), and has thousands of propositions, wholly written and demonstrated in symbols and linked together in an unbreakable chain. But the chief reason for the aloof attitude of mathematicians toward

the *Principia* seems to me to lie in the fact that the authors have admitted into the book concepts and principles based on considerations not sufficiently convincing—concepts and principles based on views opposed to those forced on mathematicians by the work of Peano, Pieri, Hilbert, Veblen, Huntington. Thus our authors have admitted into the first edition the primitive idea "assertion of an elementary function," the notion "all" as distinct from "any," the notion "function of functions" in an "intensional" sense. Thus we have an involved theory of types of propositions, with its demand that there be an infinite number of propositional logics, one for each type, instead of one logic for all propositions. Thus, also, we have the view that "the theory of propositions necessarily precedes the theory of classes" and the view that for the primitives underlying the propositional logic "the recognized methods of proving independence are not applicable."

I can perhaps make more clear the attitude toward the *Principia* of the mathematician interested in logic by making some analysis, from the latter's point of view, of the primitives for elementary propositions found in the first edition. The analysis will also hold in substance for the primitives of the revised edition. Among the primitives for elementary propositions are found the following undefined ideas: "elementary proposition," denoted by $p, q, r, \cdots$; "negation" of p, denoted by "$\sim p$"; "assertion" of p, denoted by "$\vdash p$"; "disjunction" of p and q, symbolized by "$p \vee q$". Our authors define "$p \supset q$" to mean "$\sim p \vee q$", and then give a list of primitive propositions of which the following two are types: (1) "Anything implied by a true elementary proposition is true," (2) "$\vdash : p \vee p . \supset . p$," where in (2) the dots stand for parentheses. Proposition (1), our authors observe, cannot be expressed in symbols. If we take this remark literally, then (1) says nothing about the indefinables of the system, and hence cannot be a proposition belonging to the system. Let us take it that our authors mean to say that (1) cannot be expressed wholly in symbols, that (1) means, say: (1′) "Given '$\vdash p$' and '$\vdash . p \supset q$', then '$\vdash q$'." The primitive propositions under discussion then consist of types (1′) and (2). Now what can we say of (2)? Evidently, since *all* the symbols are left undefined, *it says nothing of* p, any more than "$a + b$" says anything about a and b. Then (2) is merely a *function* of $\vdash$, p, $\vee$, and $\sim$; it is *not* a proposition. Our authors presumably mean to use "$\vdash$" and "$\supset$" as symbols of assertion; but in that case these symbols will have to be *outside* the system, in accordance with Russell's own very sound "vicious-circle principle." A proposition in our logic, then, if it is to say anything about the undefined symbols, must be of type (1′). This means that not only must any *non*-logical mathematical system use the notions and principles of general logic, but also that the logic of propositions itself must have a general logic, an unsymbolized logic, underlying it. This the authors of the *Principia* seem to overlook or to ignore when they give us propositions of type (2) above and when they regard the logic of propositions as more "fundamental" than the logic of classes.

But, it may be objected, our authors nevertheless obtain from their primitives all the facts in the theory of elementary propositions. The answer is: Not all. Nowhere among the primitive propositions or among the theorems derived from them is there found a proposition to the effect that every

elementary proposition has one and only one of the truth-values *truth* and *falsity*. A proposition such as *2·11. "$\vdash . p \vee \sim p$" may *seem* to give this fact, but in reality it does not, as is readily seen when *2·11 is written in the partly classic notation "$p \vee \sim p = 1$." We need an existence proposition to establish our proposition, and our authors' theory of types could not admit existence notions in the theory of elementary propositions!

But how about non-existence propositions derived from our primitives? If propositions of type (2) are not really propositions, how account for the host of truths that are actually obtained from such propositions? The answer here is: The authors improperly read into the symbols of their systems ideas which properly belong *outside* the system. They do this with regard to the symbols "$\vdash$" and "$\supset$". The symbol "$\vdash p$" of the *Principia* is the same as Boole's or Schröder's "$p = 1$" ("the truth-value of p is truth"), in which p and 1 are symbols of the system, but "$=$" is not. It is by means of that part of the symbol "$\vdash$" which belongs outside the system that our authors make a proposition out of the sequence of symbols (2) above. Again, "$p \supset q$" is, by definition, nothing more than "$\sim p \vee q$." It should therefore be read: "not-p or q." It must not be read, as the *Principia* does: "p implies q" nor "If p is true then q is true." Each of the last two forms is a *statement about* p and q; the first is not.

All this is not quibbling. If the distinctions made be disregarded, serious errors may result. Such an error Schröder makes when he "shows" that the duality principle for classes breaks down in the logic of propositions. And such an error our authors commit when they say (p. 121) that the analog for classes of the proposition *4·78. $\vdash :. p \supset q . \vee . p \supset r :\equiv : p . \supset q \vee r$ is false. "Put p = English people, q = men, r = women. Then p is contained in q or r, but is not contained in q and is not contained in r." The fact is that the analog for classes of *4·78 (or of any other proposition in the logic of elementary propositions) strictly holds, as may be seen by reading "$\vdash p$" as "p is the universal class 1," and by writing for "$p \supset q$" and "$p \equiv q$" only the expressions given them by definition. The error made by our authors consists in taking as the class analog of "$p \supset q$" the *statement* "p is contained in q," when the true analogue is the *function* "not-p or q." The statement "p is contained in q" is the analog of "$\vdash . \sim p \vee q$" (or "$\sim p \vee q = 1$"). Similar considerations hold for the symbol "$\equiv$".

The essence of the matter is this. The logic of propositions is simply a two-element logic of classes, an algebra of truth-values 0, 1, as I have shown elsewhere (in a paper forthcoming in the TRANSACTIONS OF THIS SOCIETY). Also, the symbols of this logic, *together with other symbols*, may be used as a *language*, a symbol language, in which all mathematical systems can be expressed, *including the propositional logic itself* (Cf. Peano's *Formulaire de Mathématiques*). As a mathematical system, the logic of propositions is amenable to the postulational treatment applicable to any other branch of mathematics. As a language, this logic has *all* its symbols *outside* the system which it expresses. This distinction between the propositional logic as a mathematical system and as a language must be made, if serious errors are to be avoided; this distinction the *Principia* does not make.

B. A. BERNSTEIN

SHORTER NOTICES

Versicherungsmathematik. By Alfred Loewy. 4th ed. Berlin, Julius Springer, 1924. 8vo. v+224 pp.

The first three editions of this book were published in the SAMMLUNG GÖSCHEN in the years 1903, 1910, and 1915, respectively. The present edition differs from the preceding chiefly in the addition of several chapters on total and permanent disability. The practice of combining total and permanent disability protection—both with regard to the waiving of premiums and the payment of annual or monthly income in the event of the insured becoming totally and permanently disabled—with the regular life insurance policy has become such a settled matter that it would seem that no book on the elements of life insurance could well afford to omit an account of it. The book contains fourteen chapters and five tables. The first two chapters deal with the elements of the theory of interest and an account of the important insurance mortality tables. A very good treatment of single premiums for life insurance and annuities is contained in the next chapter and the fourth chapter completes the theory of net premiums by developing the formulas for annual and fractional premium payments for all ordinary forms of insurance protection. In Chapter five the author deals with the subject of premium loading for expenses and contingencies and also with return premiums. In the next two chapters reserves and surplus are considered, the net level and Zillmer's preliminary term reserves receiving attention. The next two chapters are devoted to an account of joint life and survivorship annuities and select mortality tables. The construction of the latter is carefully outlined and the methods of calculating select premiums and reserves are explained. In Chapter ten the formulas connected with the subject of disabled lives are developed, including the construction of the disability mortality table and the calculation of annuities on disabled lives in terms of commutation columns. The author also devotes some pages to the consideration of selection and the formulas pertaining thereto among disabled lives. Chapter eleven goes into some detail on the construction of combined disability and mortality tables and develops the formulas for premiums and reserves on active lives for disability protection of various kinds. The remaining chapters deal with the insurance of widows and orphans, general considerations concerning reserves, and problems relating to dividends.

The first table relates to the twenty-three Deutsche Gesellschaften (the twenty-three German Life Insurance Companies Experience), and the second to the Life Table for men, German Empire, 1891–1900. Tables three and four are based on disability experience, and Table five on insurance of widows. This book has been very popular on the continent, having gone through three editions and the present edition will make it still more useful on account of the important chapters incorporated on

total and permanent disability. The student who is carried through this text will have a very good understanding of the actuarial theory underlying modern life insurance. It should be mentioned, of course, that the practice of life insurance in this country differs considerably from that in England and the continent and an American student would gain chiefly on the theoretical side from a book of this character. As an illustration of this fact I might state that all companies in this country are on the Illinois Standard reserve basis,—a subject which is not directly treated in this work.

J. W. GLOVER

Le Problème de Pappus et ses Cent Premières Solutions. By A. Maroger, with a preface by Paul Montel. Paris, 1925. viii+386 pp.

In his preface to this work Professor Montel calls attention to the fact that M. Maroger has done for a single proposition from the works of Pappus what Jean Macé (who, by the way, made a humble contribution to mathematics) did for a bite of bread in his delightful *Histoire d'une Bouchée de Pain*, and what Professor Klein did in his *Ikosaeder*. In each case the writer, starting with a simple and familiar entity, develops therefrom an elaborate treatise. He might with equal propriety have compared it with Pascal's treatment of the "arithmetic triangle" or the "mystic hexagram," or with any one of numerous other cases in which a whole theory grows from a very small seed.

This seed, in the case of the work under review, is the problem of drawing through a given point on the bisector of a right angle a line segment of given length and having its extremities on the sides of the angle. Just why this is designated as *le problème* instead of *un problème* does not appear, since it is natural to refer to "the" problem as the one which was made so well known by Descartes in the first book of *La Géométrie*. This, however, is a point of no moment; the interesting thing is to see what M. Maroger has done with the problem, namely, to give a hundred solutions out of a large number that he has found. This work has carried him into various lines of modern geometry, the results being arranged in no very systematic sequence but rather in the order in which the solutions were discovered. It is also interesting and, for the student of geometry, appetising to know that it is the author's plan to continue the study even farther. The problem, analytically considered, is one of the fourth degree, reducible to the second. As Professor Montel remarks, such cases suggest an *étoile* in a forest, so familiar to all who have tramped through such regions and have seen the numberless paths that radiate from such centers, permitting the wanderer to explore a vast region. From this problem M. Maroger has explored not only the territory of special solutions, but also various contiguous and special fields. These "digressions," as he calls them, constitute the most interesting part of his text. Students of geometry and the theory of equations will find a storehouse of interesting material in the three hundred eighty four closely printed pages of the text.

DAVID EUGENE SMITH

Sieben-und mehrstellige Tafeln der Kreis-und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion, nebst einem Anhang: Interpolations- und sonstige Formeln. By Keiichi Hayashi. Berlin, Julius Springer, 1926. vi+283 pp.

In many physical and engineering calculations, the trigonometric functions of an argument expressed in circular measure occur in combination with the hyperbolic functions of the same argument, and it is therefore very convenient to have a table giving both kinds of functions with a single entry. The work under review is far more elaborate and complete than any other tables of this kind. In the main part of Table I, x runs from 0.100 to 2.999 by intervals of 0.001, and the following columns are found on two opposite pages: x, $\phi(=x$ converted into degrees, minutes and seconds), sin x, cos x, tan x, arcsin x, arccos x, arctan x (left page); x, e^x, e^{-x}, sh x, ch x, th x, arg sh x, arg ch x (part of the table only), arg th x (right page). At the bottom of each page there is an additional table giving log nat x and the hyperbolic amplitude ϑ (where $x=\log \text{nat} \tan(\pi/4+\vartheta/2)$). The number of decimal places carried varies with the function tabulated from twelve for sin x and cos x to seven for arg sin x. The beginning of Table I is arranged somewhat differently for reasons of space, and gives the same functions to twenty places from $x=0.00000$ to 0.00100 and to ten places from $x=0.0010$ to 0.0999, the interval being one unit in the last decimal place noted. The end of Table I gives these functions from $x=3.00$ to 9.99, 10.0 to 20.0 and 21 to 50, the number of decimal places varying from 18 to 7.

Table II gives sin $(x\pi/2)$ and cos $(x\pi/2)$ from $x=0.000$ to 0.500 to ten places, and Table III e^{ξ}, $e^{-\xi}$, sh ξ, ch ξ, where $\xi=x\pi/360$, from $x=0$ to 360. Table IV gives the four products (sin, cos) x, (sh, ch)x, which form a fundamental system of the differential equation $d^4y/dx^4+y=0$ and occur, for instance, in the theory of a beam on a continuous elastic support. Tables V and X are the customary conversion tables for angular and circular measure. Table VI gives $\log_{10}\Gamma(x)$ from $x=0$ to 3 with from eight to thirteen places at intervals varying from 0.00001 to 0.01, and Table VII contains $\Gamma(x)$ to seven and eight places from -5 to $+5$ at intervals from 0.001 to 0.01. Table VIII gives the first ten powers of the integers from 1 to 100, Table IX the first 30 powers of 2, 3, 4 and 5, while finally Table XI contains factorials, their inverses, and some other series coefficients. There is an appendix, giving the main formulas for hyperbolic functions, general notes on interpolation, and special devices, with auxiliary tables, to facilitate interpolation in the main table.

One cannot but admire the author's fortitude in carrying on his work in the face of disaster. As he tells us in the preface, the manuscript of over a thousand pages was completed toward the end of 1923, only to be destroyed in a fire that consumed the engineering building of the University of Kyushu. The author immediately began his work anew, and in another two years completed the manuscript of the present tables which will certainly remain the standard of their kind for many years to come.

T. H. GRONWALL

La Géométrie des Espaces de Riemann. (MÉMORIAL DES SCIENCES MATHÉMATIQUES, No. 9.) By Élie Cartan. Paris, Gauthier-Villars, 1925. 59 pp.

This is number 9 of the series of tracts, MÉMORIAL DES SCIENCES MATHÉMATIQUES," published under the patronage of the Academy of Sciences of Paris and of several other countries.

The aim is to present in brief form the developments of Riemannian geometry and especially that which came as a consequence of Levi-Civita's notion of parallelism. Using this, the author is able to present the subject with a minimum of formal work and a maximum of geometry. It is a delight to a geometer to read this geometric argument instead of solid pages of formulas which confront one so often. The first part is in sufficient detail that one follows the argument easily and after that many of the theorems are not proved but the statement is so clear and concise that one does not feel the need of proof. The notion of covariant and contravariant systems is assumed. Chapter I presents euclidean geometry in curvilinear coordinates and discusses especially the change in coordinates of a point when the reference system with origin at M is moved into a reference system with origin at a nearby point M'. These changes are given by the formula

$$dx^i + du^i + \Gamma^i_{kr}\, X^k\, du^r = 0$$

where du^i are the coordinates of M' with respect to the coordinate system with origin at M. The left side of the formula can also be interpreted as the change of the coordinates of a moveable point or a vector. Thus

$$D_r x^i = \frac{\partial x^i}{\partial u^r} + \epsilon^i_r + \Gamma^i_{kr} x^k, \quad \epsilon^i_r = 1 \text{ if } i = r, \quad \epsilon^i_r = 0, \text{ if } i \neq r$$

$$D_r X^i = \frac{\partial X^i}{\partial u^r} + \Gamma^i_{kr} X^k$$

where x^r are the coordinates of a point and X^i of a vector. The Γ's are then shown to be Christoffel symbols of the second kind.

Chapter II then discusses Riemannian space and shows that an osculating euclidean space exists at each point, which will contain two near by reference systems and hence the above formulas can be applied to this space. This leads to the Levi-Civita parallelism and the rest of the booklet is an application of these notions.

To give a more definite notion of the subjects treated I give the titles of the chapters. Chapter I, curvilinear coordinates in euclidean geometry; Chapter II, Riemannian spaces; Chapter III, Riemann-Christoffel tensor and Riemannian curvature; Chapter IV, Bianchi's identities, vector curvature, spaces of constant curvature; Chapter V, totally geodesic varieties; Chapter VI, varieties immersed in a Riemannian space; Chapter VII, class, degrees of freedom, holonomic groups.

C. L. E. MOORE

Algebraische Flächen. By H. W. E. Jung. Hannover, Helwigsche Verlagsbuchhandlung, 1925. xvi+410 pp.

This is the logical continuation of the author's recent volume on algebraic curves and is directly in line with the well known treatise of Hensel-Landsberg. It contains in essence his investigations for the last two decades, with a surprising blank at the very outset. The whole edifice rests on the possibility of representing the vicinity of any point on an analytic surface by means of a finite number of sets of power series in two variables. The proofs as now found in the literature (Black, Hensel, and Jung—the best is due to the first named author and is found in PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 37 (1902)) are, however, long and unwieldy, so that the author contents himself with assuming this all-important proposition without proof. This calls for comments that we shall reserve for another occasion.

It will be remembered that the positions on a Riemann surface are treated by Hensel, Landsberg, and Jung as arithmetical divisors. At bottom the associated symbolical operations are in no sense different from those that occur in connection with the Noether-Brill theory of groups of points, elements being merely multiplied instead of added. A similar remark applies to the author's painstakingly defined divisors on an algebraic surface, where things naturally do not go quite so smoothly. His theory, by the way, can be developed with much greater ease by means of some rather simple considerations of a topological nature. Be that as it may Jung is mainly interested in the projective group; hence his book, which concerns itself essentially with certain intersection numbers and projective invariants, is much more in line with the school of Cayley and Sylvester than with the work of modern algebraic geometers. Of the invariant integers which play such an important part in their work that of Zeuthen-Segre alone receives much attention from him. In a treatise published at the present time we would have enjoyed seeing a more varied selection of topics. However, within the narrow scope deliberately chosen by the author, he has done well indeed and geometers will read him with interest and profit.

S. LEFSCHETZ

Die mathematischen Hilfsmittel des Physikers. By Erwin Madelung. Second edition. Berlin, Julius Springer, 1925. xiii+283 pp.

This monograph constitutes the fourth volume of the collection edited by R. Courant and entitled DIE GRUNDLEHREN DER MATHEMATISCHEN WISSENSCHAFTEN IN EINZELDARSTELLUNG. It contains statements of all the fundamental principles and outlines of all the methods which, in the author's opinion, may be required by the theoretical or computing physicist. In the chapters which deal with pure mathematics practically everything is included that has been used heretofore in the solution of physical problems, as well as additional material that gives promise of future applicability. Of course, the territory of the common tables of integrals, numerical data, etc., is not encroached upon.

Proofs are indicated only in a few places. With these exceptions, the presentation is consistently limited to definitions of the fundamental concepts, to the associated formulations, and to the final results. Fortunately especial stress is laid on the algebra of vectors and tensors.

The first ten chapters are devoted to pure mathematics as may be inferred from the following chief headings: algebra, differential and integral calculus, series, functions, transformations, differential equations, linear integral equations, calculus of variations, theory of probability, and vector analysis. The remaining chapters and the appendix really constitute an outline of a valuable course in theoretical physics. More specificially, the fields represented are mechanics, electricity, theory of relativity, thermodynamics, quantum theory, and a few physical applications of the calculus of probability. A short list of authoritative references is appended to each chapter.

The second edition differs from the first principally in certain additions such as boundary problems, tensor analysis, theory of the top, the general relativity theory, and the quantum theory. The printing is very clear, the spacing is adequate, and typographical errors seem to be absent. In the reviewer's opinion this book merits the attention of all lecturers on applied mathematics and a working knowledge of its contents should be a minimum requirement in mathematics of all candidates for the degree of doctor of philosophy in physics.

H. S. UHLER

Binomial Factorizations, Vol. V. By A. J. C. Cunningham. London, Francis Hodgson, 1924. lxxii+120 pp.

This volume is one of a set of seven giving the smallest roots of the congruence $y^n \pm 1 \equiv 0 \pmod{p}$ with p running through the available primes (of the form $2nx+1$) to various high limits. From these tables the author builds up factorization tables for different values of y. Thus from the table of roots of $y^{11}-1 \equiv 0 \pmod{89}$, namely: 2, 4, 8, 16, 32, 39, 45, 64, 67, 78, he is able to insert a factor 89 in the list of divisors of what he calls "Simple Undecimans" (that is to say in the divisors of $(y^{11}-1)/(y-1)$) opposite the corresponding values of y. This table of "Simple Undecimans" runs to $y=100$, and contains only 36 complete factorizations. To get these factorizations over nine hundred congruences had to be solved and over nine thousand roots listed. The corresponding table $n=13$ gives only 26 complete factorizations in the first hundred "Simple Tredecimans" as the result of solving some 800 congruences and of listing some 9,600 roots. The showing for "Simple Septimans" ($n=7$) given in previous volumes is not much better. However, although the percentage of complete factorizations is not impressively large, nevertheless the incomplete factorization of these high numbers will serve to settle the question of their primality and often this is all that is desired. Moreover the determination of one factor will make the complete factorization by other methods easier to perform.

D. N. LEHMER

Factorization of $y^n \pm 1$, $y = 2, 3, 5, 6, 7, 10, 11, 12$ *up to high powers* (n). By A. J. C. Cunningham and H. J. Woodall. London, Francis Hodgson, 1925. xx+24 pp.

This little volume gives the latest account of the efforts of arithmeticians in factoring numbers of the form $y^n \pm 1$. The tables for $2^n \pm 1$ and for $10^n \pm 1$ owe much to other computers but the remainder of the work is the outcome of some thirty years of faithful work on the part of these veteran computers. The table for $2^n - 1$ runs up to $n = 500$ but the blank spaces for n greater than 100 remind one of the unexplored regions in ancient maps. No doubt in the next few years explorers armed with high-power computing machines will fill up many of these white spaces. Within the last half year $2^{139} - 1$ has been proved composite and before the end of the year Kraitchik's investigation concerning $2^{257} - 1$ will be confirmed or rejected by a recomputation. Every year witnesses the invention of some new device for examining the primality of "large numbers" and doubtless what today are considered as "large numbers" will be "small numbers" for the next generation. In the meantime this little book will serve to point out the gaps.

D. N. LEHMER

Atomicity and Quanta. Being the Rouse Ball Lecture delivered on May 11, 1925. By J. H. Jeans. Cambridge University Press, 1926.

In this lecture, Dr. Jeans has sketched the development, and outlined the present state of the quantum-theory. As space and time are linked together in the theory of relativity, so are matter and energy; and the atomicity of matter may imply a like property in energy. The evidence for the atomicity of energy furnished by recent experimental results seems, at first sight, almost conclusive; and the difficulty of reconciling a bullet-theory of radiation with the undulatory theory, which appears to be essential for accounting for the greater part of optics, leads to the conclusion that "we are still very far from understanding the working of the atom or the true meaning of atomicity and quanta."

The lecture is written in the lucid style characteristic of the distinguished author, and every one interested in modern physics will be well repaid for reading it.

E. P. ADAMS

NOTES

At the Summer Meeting of the Society in Columbus, the Council of the Society authorized the officers of the Society to enter into negotiations with the authorities of the Johns Hopkins University with a view toward the participation of the Society in the conduct and publication of the AMERICAN JOURNAL OF MATHEMATICS. These negotiations have progressed favorably, and it is hoped that a final arrangement may be announced at the time of the Christmas Meeting in Philadelphia. Although both the TRANSACTIONS and the BULLETIN have been increased in size to approximately 800 pages each, the volume of mathematical papers suitable for publication has so increased that further means of publication must be sought, and an increase in the size of the AMERICAN JOURNAL, which may be realized through the aid of the Society, seems a suitable and a desirable means to this end.

The List of Officers and Members of the American Mathematical Society, 1926, was published as a supplement to the September-October issue of this BULLETIN (vol. 32, No. 5). In addition to the usual lists, it contains a *Historical Note* (pp. 6–7) that places on record for the first time some of the details concerning the formation of the Society and some of the more significant events of its history. The supplement contains also an account of the Prize Funds of the Society, the By-Laws of the Society, and the reports of the Treasurer for 1924 and 1925.

The third number of volume 28 of the TRANSACTIONS OF THIS SOCIETY (July, 1926) contains the following papers: *Double binary forms with the closure property*, by A. B. Coble; *On a class of polynomials in the theory of Bessel's functions*, by J. H. McDonald; *On the zeros of the function* $\beta(s)$ *associated with the gamma function*, by T. H. Gronwall; *Osculating curves and surfaces*, by P. Franklin; *On Laplace's integral equations*, by J. D. Tamarkin; *On Volterra's integro-functional equation*, by J. D. Tamarkin; *On the discriminant of ternary forms and a certain class of surfaces*, by A. Emch; *On the convergence of certain methods of closest approximation*, by E. Carlson; *Functions of plurisegments*, by A. J. Maria; *Sets of postulates for the logic of propositions*, by B. A. Bernstein; *Multiple sheeted spaces and manifolds of states of motion*, by H. Hotelling; *A theory of a general net on a surface*, by V. G. Grove; *Cubic curves and desmic surfaces*, by R. M. Mathews; *Expansion problems in connection with homogeneous linear q-difference equations*, by M. G. Carman; *Some properties of limited continua, irreducible between two points*, by W. A. Wilson; *Application of the theory of relative cyclic fields to both cases of Fermat's last theorem*, by H. S. Vandiver.

The third number of volume 48 of the AMERICAN JOURNAL OF MATHEMATICS (July, 1926) contains: *On the structure of a continuum, limited and irreducible between two points*, by W. A. Wilson; *The expansion problem*

for a certain system of ordinary linear second order differential equations, by M. G. Carman; *The distribution of the lift over thin wing sections*, by C. A. Shook; *The correspondence between the tangent plane of a surface and its point of contact*, by E. P. Lane; *On multiple iterated integrals*, by H. J. Ettlinger; *Cubic involutions and a* C_3, by H. S. White.

The concluding number of volume 27, series 2, of the ANNALS OF MATHEMATICS (September, 1926) contains: *On classes of ideals in a quadratic field*, by H. H. Mitchell; *Developments in Legendre polynomials*, by M. H. Stone; *On certain two-dimensional varieties of order* $r(r-1)/2$ *in r-space and certain curves traced on them*, by B. C. Wong; *On class number relations implied by representations as sums of an odd number of squares*, by E. T. Bell; *Table of quadratic residues*, by A. A. Bennett; *Anharmonic groups*, by A. S. Hathaway; *Some general determinant theorems in tensor notation*, by C. M. Cramlet; *Some conditions under which a continuum is a continuous curve*, by H. M. Gehman; *Successive derivatives of a function of several functions*, by L. S. Dederick; *Plane cubic curves in the Galois field of order 2*, by A. D. Campbell; *On certain determinant relations*, by W. H. Metzler; *On the decomposition of tensors*, by J. W. Alexander; *Flow in a Möbius strip*, by B. O. Koopman; *A class of reciprocal functions*, by E. Hille; *Some extensions of the generalized Kronecker symbol*, by J. J. Nassau; *On the indeterminate equation* $t^2-p^2Du^2=1$ by D. H. Lehmer; *The operations of Boolean algebras*, by O. Frink; *A proof of Petersen's theorem*, by O. Frink; *Geometry of a set of n functions*, by L. Ingold; *Irregular fields*, by E. T. Bell; *On the approximate solution of the integro-differential equations of mathematical physics*, by N. Kryloff; *A new proof of Parseval's identity for trigonometric functions*, by J. Tamarkin; *Tensors whose components are absolute constants*, by T. Y. Thomas; *Some convergence proofs in the vector analysis of function space*, by D. Jackson; *Harmonic cubics*, by T. R. Hollcroft.

It has been decided to hold the next International Mathematical Congress at Bologna, in 1928.

The British Association for the Advancement of Science held its summer meeting of 1926 at Oxford, under the presidency of the Prince of Wales. On this occasion the University of Oxford conferred honorary doctorates on Professor Niels Bohr, Sir Frank W. Dyson, and Professors A. S. Eddington, Vito Volterra, and Wilhelm Wien. The Association will hold its next summer meeting at Leeds, August 31–September 7, 1927.

Cambridge University has awarded its John Winbolt prize jointly to H. Bateman, of Trinity College, and R. J. Smith, of St. John's College, for a theoretical investigation of some elastic problems in thin rectangular plates.

The New York Academy of Sciences offers a prize (the A. Cressy Morrison Prize) to be awarded in December, 1926, for an essay on the intra-atomic activity of the sun.

Recent elections to the section of mathematics of the Reale Accademia dei Lincei are the following: Professor F. Severi, of the University of

Rome, to national membership; Professor G. Scorza, of the University of Naples, to corresponding membership; Professor E. Cartan, of the University of Paris, to foreign membership.

Professor G. Fubini, of the University of Turin, has been elected a fellow of the Italian Society of Sciences.

Professors E. Laura, of the University of Padua, and A. dell'Agnola, of the Higher Commercial School of Venice, have been elected corresponding members of the Reale Istituto Veneto.

Professor A. Palatini, of the University of Pavia, has been elected a corresponding member of the Reale Istituto Lombardo.

Professor Peter Debye, of the Zurich Technical School, has been elected a member of the Danish Academy of Sciences.

Professor Arnold Sommerfeld has been elected an honorary member of the Manchester Literary and Philosophical Society.

Kenyon College has conferred the honorary degree of doctor of laws on Professor M. I. Pupin, of Columbia University.

The Institute of International Education (522 Fifth Avenue, New York City) will award a certain number of fellowships for the year 1927–28 for study in France (American Field Service Fellowships) and for study in Germany (American German Student Exchange Fellowships). For the French fellowships, applications should be addressed to Dr. Stephen P. Duggan, at the Institute, and should reach him by January 1, 1927; for the German fellowships, applications should be addressed to Mr. Carl J. Friedrich, at the Institute, and should reach him by February 15, 1927.

A mathematical institute has recently been endowed at the University of Jerusalem; it will be under the general direction of Professor E. Landau, of Göttingen. The Institute has acquired the library of the late Professor Felix Klein. Dr. B. Amira, of the University of Geneva, will be in charge of the courses in differential and integral calculus and in higher mathematics for students of chemistry.

Johns Hopkins University celebrated its fiftieth anniversary on October 22–23, 1926. Among the alumni who delivered addresses on this occasion were Professors A. B. Coble, of the University of Illinois, and L. P. Eisenhart, of Princeton University.

The following distinguished European mathematical physicists delivered lectures in various American universities in the fall of 1926: Professor H. A. Lorentz, of the University of Leyden, as visiting lecturer on the Schiff foundation; Professor Hermann Weyl, of the Zurich Technical School; Professor Theodor von Karmán, of the Aerodynamical Laboratories at Aachen, under the auspices of the Daniel Guggenheim Fund for the Promotion of Research in Aeronautics.

Dr. Albert Betz, of the University of Göttingen, has been promoted to an associate professorship of applied mathematics.

Dr. Robert Furch, of the Karlsruhe Technical School, has been appointed to an associate professorship of mathematics at the University of Rostock.

Professor P. Koebe, of the University of Jena, has been appointed professor of mathematics at the University of Leipzig.

Dr. Arpad Nadori, of the University of Göttingen, has been promoted to an associate professorship of mathematics.

Professor Ludwig Neder, of the University of Tübingen, has been appointed professor of mathematics at the University of Münster.

Professor Max Planck, of the University of Berlin, has retired.

Dr. Ludwig Schiller, of the University of Leipzig, has been promoted to an associate professorship of applied mathematics and thermodynamics.

Dr. Adolf Schlötzer, of the Munich Technical School, has been appointed professor of practical geometry and geodesy at the Karlsruhe Technical School.

Dr. Erich Schönberg, of the University of Breslau, has been promoted to an associate professorship of mathematics and astronomy.

Associate Professor Gabriel Szegö, of the University of Berlin, has been appointed professor of mathematics at the University of Königsberg.

Dr. Georg Wentzel, of the University of Munich, has been appointed associate professor of mathematical physics at the University of Leipzig.

Dr. E. Zermelo, professor emeritus of the University of Zurich, has been given the title of honorary professor at the University of Freiburg i. Br.

The following have been admitted as privat docents: Dr. Heinrich Behmann, at the University of Halle; Dr. Walther Mayer, at the University of Vienna.

Dr. Lucien Godeaux has been appointed professor of mathematics at the University of Liége, as successor to the late Professor J. Fairon.

Mr. A. E. Ingham, of Trinity College, Cambridge, has been appointed reader in mathematical analysis at the University of Leeds.

Professor L. D. Ames, of the Texas Technological College, has been appointed professor of mathematics at the University of Southern California.

Mr. C. F. Barr, of Purdue University, has been appointed assistant professor of mathematics at the University of Wyoming.

Dr. E. M. Berry, of Purdue University, has been appointed head of the department of mathematics at Lynchburg College.

Mr. E. C. Bower, of the United States Naval Observatory, has been appointed assistant professor of mathematics and astronomy at Ohio Wesleyan University.

Associate Professor Abraham Cohen, of Johns Hopkins University, has been promoted to a full professorship of mathematics.

Associate Professor H. B. Curtis, of Marquette University, has been appointed head of the department of mathematics at Lake Forest College.

Mr. D. C. Harkin has been appointed lecturer in mathematics at the University of Manitoba.

At Cornell University, Professor W. A. Hurwitz has been made chairman of the department of mathematics.

Miss Myra I. Johnson has been appointed professor of mathematics at Blackstone College for Women.

Mr. G. A. Lyle, of Lehigh University, has been appointed adjunct professor of mathematics at the Texas Technological College.

President G. D. Olds, of Amherst College, has resigned; he will continue in office until his successor has been appointed. President Olds was for many years professor of mathematics at Amherst.

Associate Professor H. R. Phalen, of the Armour Institute of Technology, has been appointed professor of mathematics at St. Stephen's College.

At the University of Wyoming, Professor O. H. Rechard has been made head of the department of mathematics.

Professor H. L. Rietz, of the University of Iowa, has been appointed actuary of the Pension Commission of Chicago, to investigate the pension systems of the city.

Professor L. V. Robinson, of Oklahoma City University, has been appointed adjunct professor of mathematics at the Texas Technological College.

Professor S. A. Rowland, of Union College, has been appointed professor of mathematics at Ohio Wesleyan University.

Associate Professor Mary E. Sinclair, of Oberlin College, has been promoted to a full professorship of mathematics.

Assistant Professor H. L. Smith, of the University of Minnesota, has been appointed assistant professor of mathematics at Louisiana State University.

Associate Professor F. W. Sparks, of Louisiana State Normal College, has been appointed associate professor of mathematics at the Texas Technological College.

Assistant Professor J. S. Turner, of Iowa State College, has been promoted to an associate professorship of mathematics.

Mr. A. H. Wait, of the University of Wisconsin, has been appointed adjunct professor of mathematics at the Texas Technological College.

The following appointments as instructor in mathematics are announced: Cornell University, Mr. A. G. Montgomery and Mr. H. L. Schug; Hunter College, Miss Laura Guggenbühl; Lehigh University, Mr. L. J. Paradiso and Mr. G. W. Riddle; Montana State School of Mines, Mr. Alexander Maslow; University of Nebraska, Mr. H. P. Doole; University of New Hampshire, Mr. M. R. Solt; New York University (Washington Square College), Miss Elizabeth Berger; Princeton University, Mr. M. S. Knebelman; Texas Technological College, Mr. P. K. Rees.

The death is announced of Dr. Alexander Beck-Ustni, formerly professor of astronomy and mathematics at the Riga Technical School.

The death is announced of Professor A. von Sourek, of the University of Sofia.

The death is announced of Professor Theodor Des Coudres, of the department of theoretical physics of the University of Leipzig.

Professor Adolf Krazer, of the Karlsruhe Technical School, died August 7, 1926, at the age of sixty-eight.

The death is announced of Dr. Bernhard Pattenhausen, formerly professor of geodesy and mathematics at the Dresden Technical School.

Dr. J. L. E. Dreyer, formerly director of Armagh Observatory, Ireland, died September 14, 1926, at the age of seventy-four. He was known for his work in the history of astronomy.

Sir Peter R. S. Lang, emeritus professor of mathematics at the University of St. Andrews, died July 5, 1926, at the age of seventy-five.

Dr. H. A. Howe, dean of the College of Liberal Arts of the University of Denver, and director of the Chamberlin Observatory of the University has died at the age of sixty-seven.

Professor W. J. Hussey, director of the observatory of the University of Michigan, died in London, October 28, 1926, at the age of sixty-four. Professor Hussey was en route to Bloemfontein, South Africa, to assist in the establishment of a university observatory station there.

Dr. F. E. Nipher, professor emeritus of physics at Washington University, died October 6, 1926, at the age of seventy-eight.

Mr. P. C. H. Papps, of the Mutual Benefit Life Insurance Company, Newark, died May 18, 1926. Mr. Papps was a member of the American Mathematical Society.

Professor G. H. Scott, dean of Illinois College, and formerly professor of mathematics at Yankton College, died September 12, 1926.

Dr. C. A. Waldo, professor emeritus at Washington University, died October 1, 1926, at the age of seventy-four.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APPELL (P.). Henri Poincaré. Paris, Plon, 1925. 120 pp.

APPELL (P.) et KAMPÉ DE FÉRIET (J.). Fonctions hypergéométriques et hypersphériques. Polynomes d'Hermite. Paris, Gauthier-Villars, 1926.

ARISTOTLE. The works of Aristotle translated into English under the editorship of W. D. Ross. Categoriae and De interpretatione, by E. M. Edghill; Analytica priora, by A. J. Jenkinson; Analytica posteriora, by G. R. G. Mure. Oxford, Clarendon Press, 1926. 348 pp.

ARNDT (J. G. A.). Über die Darstellung ganzer Zahlen als Summen von sieben Kuben. (Dissertation.) Göttingen, 1925.

BÄUMLER (A.). See WEYL (H.).

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F. S. Nowlan, *Representation of integers by certain ternary cubic forms.*

Page 376, formula (5): the sign of the third term on the right side ($5\alpha_6^2\alpha_2$), and the sign of the seventh term (α_1^3) should be plus.

Page 378, lines 11–12: The expression for F_1 should agree with the corrected form of formula (5), p. 376; on p. 378 the sign of the last term should be plus.

T. H. Hildebrandt, *The Borel Theorem and its generalizations.*

Page 453, second footnote: omit the last sentence, beginning "Moore's example, however, is · · · ".